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General covariance of the non-abelian DBI-action

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Abstract: In this paper we study the action for N D0-branes in a curved background. In particular, we focus on the meaning of space-time diffeomorphism invariance. For a single D-brane, diffeomorphism invariance acts in a naive way on the world-volume fields, but for multiple D-branes, the meaning of diffeomorphism invariance is much more obscure. The problem goes beyond the determination of an ordering of the U(N)-valued fields, because one can show that there is no lift of ordinary diffeomorphisms to matrix-valued diffeomorphisms. On the other hand, the action can presumably be constructed from perturbative string theory calculations. Based on the general characteristics of such calculations we determine a set of constraints on the action for N D0-branes, that ensure space-time covariance. These constraints can be solved order by order, but they are insufficient to determine the action completely. All solutions to the constraints obey the axioms of D-geometry. Moreover the action must contain new terms. This exhibits clearly that the answer is more than a suitable ordering of the action of a single D0 brane.

Keywords: Sigma Models, D-branes, M(atrix) Theories, Non-Commutative Geometry.

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1. Introduction

In string theory there is no invariant notion of space-time geometry, rather it depends in a non-trivial way on the object that is being used to probe the geometry. In particular, closed strings see a different geometry from the one seen by open strings or the one seen by D-branes. As explained in [1], the metric seen by a single D0-brane is a well-defined quantity and is referred to by the term 'D-geometry'. It is an interesting question what the geometry is that is seen by N coincident D-branes. The flat-space action for N coincident D-branes is given by the dimensional reduction of $N = 1$ $d = 10$ $\mathrm{U}(N)$ Super-Yang-Mills, so already in this case something non-trivial happens. The adjoint scalar fields now act as probes of the surrounding geometry; as space-time coordinates have been replaced by $N \times N$ matrices, it suggests that space-time has become non-commutative [2].

The generalization of this action to curved space is the problem we wish to address in this paper. This is a special case of the more general problem to find the action for D-branes in arbitrary closed string backgrounds. In the presence of e.g. a closed string NS-NS B-field, the action for D-branes remains similar except that the ordinary product of fields is replaced by a non-commutative $*$-product [3]. Thus, a non-zero B-field gives rise to new geometrical structures on the D-brane world volume. The presence of transversal RR fields gives rise to several interesting physical phenomena, such as the Myers effect [4], which resolves classical supergravity singularities. One may expect that similar phenomena will appear in the description of D-branes in curved space, i.e. that on the one hand some new non-commutative generalization of curved space will appear, and that on the other hand we will find a purely geometrical version of the Myers effect.

The approach we will take is to study the notion of diffeomorphism invariance for D0-branes. If we know how the world-volume fields transform under space-time diffeomorphisms, the action in a curved background is completely determined, as the graviton is the gauge field enforcing local coordinate invariance. In particular, the kinetic term in the action of a single D0 brane, $S \sim \int dt\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, is completely fixed by diffeomorphism invariance. If we replace the fields x by $N \times N$ matrices, we can write down a similar action where the metric is replaced by a matrix-valued metric, and the group of diffeomorphisms is replaced by a group of matrix-valued diffeomorphisms. The matrix-valued metric should be a functional of the original metric, since both actions are derived in the same closed string background. In particular, since the closed string background is diffeomorphism invariant, the same group of diffeomorphisms should be represented both in the action of a single as well as in the action for multiple D-branes. On the other hand, we will show that there is no lift of the diffeomorphism group into the group of matrix-valued diffeomorphisms. It follows that a new symmetry is needed to impose diffeomorphism invariance for matrix-valued fields. We will propose such a symmetry and explore its consequences. Though requirement of diffeomorphism invariance considerably constrains the action for matrix-valued fields, it does not uniquely determine it. This non-uniqueness is consistent with the results of [5, 6], where different results for the stress-tensors of D-branes in bosonic and superstring theory were obtained.

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The action for D-branes in curved space has been considered before by various authors. In [1, 7] the idea was to find an ordering of the answer for a single D-brane that has the property that the masses of off-diagonal fluctuations of the matrix-valued fields have masses proportional to the geodesic length between separated D-branes. In [8, 9] the linear coupling to the background metric was derived using the matrix theory interpretation [10] of the action of N D0-branes. The linear coupling to the background metric can also be derived directly using worldsheet methods and is related to the stress-tensor of non-commutative gauge theories [6, 11, 12]. Alternatively, one can try to use the connection with matrix theory to try to determine the coupling [13]. Through T-duality the D0-brane action is related to the non-abelian Born-Infeld action. A promising approach here is the matching of its BPS spectrum with that of string theory [14]. Symmetry principles have also been used e.g. in [15] in an attempt to find the ordering of the terms that are higher order in the gauge field field strength. For some specific curved backgrounds, one is able to explicitly construct the action, see [16] amongst others.

The outline of this paper is as follows. In section 2 we give a precise formulation of the problem. We show that the linear results of [8, 9] are invariant under space-time diffeomorphisms up to terms of higher order. In section 3 we show that the group of matrix-valued diffeomorphisms does not contain regular diffeomorphisms in a straightforward way. Using input from open-string scattering amplitudes we reformulate the full non-linear problem into a set of consistency conditions on the manifestly covariant normal coordinate version of the theory. In section 4 we give the solution to these constraints through sixth order in the fields and show the necessity of novel vertices. In section 5 we discuss some properties of the action. We propose a curved-space generalization of the $\mathrm{Tr}([X^i, X^j]^2)$ potential in terms of the matrix-valued metric that appears in the kinetic term. With this potential, which is covariant and reproduces the linearized coupling of [8, 9], the action automatically fulfills the axioms of D-geometry put forth by Douglas. Conclusions are given in section 6, and some details that are omitted from the main text are given in the appendices.

We will adhere to the convention that all upper-case "coordinates" $X, Y, Z, \ldots$ are matrix-valued and do not commute, whereas lower-case "coordinates" $x, y, z, \ldots$ are regular commuting ones. Roman indices $i, k, \ell, \ldots$ indicate directions transverse to the D-brane, Greek indices $\mu, \nu, \ldots$ target-space directions (worldvolume plus transverse) and the Greek indices from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots$ are $\mathrm{U}(N)$ matrix-labels.

2. The problem

2.1 Covariance and particle-geometry

As we briefly mentioned in the introduction, the effective action for the zero-modes of a single Dp-brane in flat space, up to second order in derivatives, is given by the reduction of $N = 1$ $d = 10$ abelian super-Yang-Mills to $p+1$ dimensions. The scalars obtained from the dimensional reduction parametrize the transverse motion. For a D0-brane the action — a gauged linear sigma model — is therefore simply that of a point particle in nine euclidean

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dimensions upon elimination of the non-dynamical U(1) gauge field.

$$S_{D0}^{\text{free}} = -\frac{T_0}{2}\int d\tau D_\tau x^i D_\tau x_i \qquad \xrightarrow{A_\tau=0} \qquad S_{D0}^{\text{free}} = -\frac{T_0}{2}\int d\tau \dot{x}^i \dot{x}_i \tag{2.1}$$

with $D_\tau = \partial/\partial\tau + A_\tau$.

Requiring in addition that the action is invariant under coordinate changes $\delta x^i = \xi^i(x)$ in the nine transverse directions, yields, of course, the action for a massive particle in a curved background with mass T_0.

$$S_{D0} = -\frac{T_0}{2}\int d\tau g_{ij}(x)\dot{x}^i\dot{x}^j\,. \tag{2.2}$$

Let us briefly recall the details why.

With a local infinitesimal diffeomorphism

$$\delta x^i = \xi^i(x) \tag{2.3}$$

one derives the gravitational current, the stress tensor,

$$T^{ij}(x) = \frac{T_0}{2}\int d\tau \delta^D(x - x(\tau))\dot{x}^i\dot{x}^j\,. \tag{2.4}$$

It is conserved on shell

$$\partial_i T^{ij}(x) = \frac{T_0}{2}\int d\tau \delta^D(x - x(\tau))\ddot{x}^j = -\int d\tau \delta^D(x - x(\tau))\frac{\delta S_{D0}^{\text{free}}}{\delta x_j} \tag{2.5}$$

for the flat-space particle action. The stress tensor thus couples naturally to the linearized gauge-field, the graviton $h_{ij} = g_{ij} - \eta_{ij}$.

$$S_{D0} = S_{D0}^{\text{free}} + \int d^D x T^{ij}(x)h_{ij}(x)\,. \tag{2.6}$$

The action (2.6) is to first order invariant under linearized gauge transformations (diffeomorphisms) where the graviton transforms into the derivative of the local gauge parameter.

$$\begin{aligned}\delta x^i &= \xi^i(x)\\ \delta h_{ij} &= -\partial_{(i}\xi_{j)}\\ \delta S_{D0} &= 0 + \mathcal{O}\left(\xi^2, \xi h, h^2\right).\end{aligned} \tag{2.7}$$

To find an action invariant to all orders, one uses e.g. the Noether method. One finds one only needs to adjust the transformation rules of h_{ij} to include non-linear terms, to the standard transformations of a metric

$$\delta(\eta_{ij} + h_{ij}) \equiv \delta g_{ij} = -D_{(i}\xi_{j)} \tag{2.8}$$

with D_i the covariant derivative with the Christoffel connection of g_{ij}. There is no need to change the transformation rules of x nor to add additional couplings to the action.

As the integrations in (2.6) are over two different sets of parameters it is more convenient to think of the stress tensor in terms of moments

$$\begin{aligned}\int d^D x T^{ij}(x) h_{ij}(x) &= \sum_{n=0}^{\infty} \int \frac{d^D x}{n!} T^{ij}(x) x^{k_1} \cdots x^{k_n} \partial_{k_1} \cdots \partial_{k_n} h_{ij}(0) \\ &= \sum_{n=0}^{\infty} \int \frac{d\tau}{n!} T^{ij(k_1 \cdots k_n)}(x(\tau)) \partial_{k_1} \cdots \partial_{k_n} h_{ij}(0) \,. \end{aligned} \tag{2.9}$$

In the last step we have used the delta function in (2.4). This clarifies that in quantum field theory terms the particle action is better understood as belonging to an infinite set of equivalent theories related by the (coupled global) "symmetry" transformations on the fields x^i and couplings g_{ij}

$$\begin{aligned} Z(g_{ij}) = \int \mathcal{D}x \exp(S_{\rm part}[x^i, g_{ij}]) &\simeq \int \mathcal{D}(x+\delta x) \exp(S_{\rm part}[x^i + \delta x^i, g_{ij} + \delta g_{ij}]) \\ \delta x^i &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{k_1} \cdots x^{k_n} \partial_{k_1} \cdots \partial_{k_n} \xi^i(0) \\ \delta \partial_{k_1} \ldots \partial_{k_n} h_{ij}(0) &= -\sum_{n=0}^{\infty} \partial_{k_1} \ldots \partial_{k_n} \partial_{(i} \xi_{j)}(0) \,. \end{aligned} \tag{2.10}$$

Except for the "symmetries" with $n = 0, 1$ all of these are non-linear.

2.2 Formulation of the problem: covariance and D-geometry

For N D0-branes, superposed in flat space, the gauge group enhances to $\mathrm{U}(N)$ and becomes non-abelian [2]. The scalars $x^i(\tau)$ become $N \times N$ matrices transforming in the adjoint. The immediate connection with the location of the D-brane in space-time is therefore lost. Only to the (diagonal) expectation values can we ascribe such a meaning. The question we wish to answer is how to couple this theory to a curved background.

The obvious generalization of (2.2) is

$$S_{D0} = -\frac{T_0}{2} \int d\tau \, G_{i\alpha\beta\, j\gamma\delta}(X) \dot{X}^{i\alpha\beta} \dot{X}^{j\gamma\delta} \,, \tag{2.11}$$

which is the most general action one can write down to second order in the derivatives. We included explicitly the matrix indices $\alpha, \beta, \gamma, \delta$ in the action. It is clear that this action is invariant under general "matrix-valued" diffeomorphisms, *if* we view the triple $i\alpha\beta$ as a single index and we let the metric transform in the usual way. Eq. (2.11) is nothing but an unconventional form of the sigma model action for dN^2 degrees of freedom ($i = 1 \ldots d$; $\alpha, \beta = 1 \ldots N$) and the starting point of D-geometry [1]. In section 4.4 we will take this action as a starting point to construct a diffeomorphism invariant effective action for N D-branes in d dimensions.

What are the further requirements the action (2.11) must fulfill (cf. the axioms of D-geometry [1])? First of all, the action can be derived from string theory disc diagrams with arbitrarily many graviton vertex operators in the interior and scalar field vertex operators carrying Chan-Paton factors on the boundary. Hence the action should consist of a single-trace [17]. In other words, if we would expand the metric $G_{i\alpha\beta,j\gamma\delta}(X)$ around $X^i = 0$ in a power series, the action should be a sum of traces of polynomials in X^i.

A second requirement, which is also obvious from string theory, is that if we take the matrices X^i to be diagonal, the action should reduce to a sum of (2.1) for each diagonal entry, i.e. a sum of N actions for a single D0 brane.

The third requirement of [1] is that if we expand the action around a diagonal configuration for X^i, the masses of the off-diagonal fluctuations should be proportional to the geodesic distance between the points given by the diagonal values of X^i. This assumes that the off-diagonal elements of X^i correspond to the vertex operators that create open strings between separated D-branes. These masses are independent of field redefinitions that preserve the single trace structure of the action, and will therefore be diffeomorphism invariant. To determine the masses we need the curved space version of the potential term $\mathrm{Tr}([X,X]^2)$ that is part of the full non-abelian D0-brane action. We postpone the construction of the potential and the calculation of the masses to section 5.

The fourth axiom of D-geometry: that the classical moduli space is the symmetric product $\mathcal{M}^N/S_N$ of N copies of the transverse space $\mathcal{M}$ also depends on the exact form of the potential and will be discussed in this section as well.

The main important new ingredient is that we will explicitly impose diffeomorphism invariance, in addition to the four axioms of [1]. To state this requirement, we need to remember that the metric in (2.11) is not arbitrary but a functional of the closed string metric g_{ij}. In other words, the action (2.11) is really an functional of g_{ij} and X^i,

$$S \equiv S[g_{ij}, X^j]\,. \tag{2.12}$$

Now suppose that g and g' are two metrics related by a target space diffeomorphism. Physics should not be sensitive to the difference between g and g', and therefore there should exist a field redefinition $X \to X'(x)$ such that

$$S[g,X] = S[g',X']\,. \tag{2.13}$$

As we will show, this is in fact a highly non-trivial requirement.

Finally we will also impose a sixth requirement that the linearized coupling of the graviton in the action agrees with the coupling proposed in [8, 9], and which has been confirmed by world-sheet calculations in [6]. The result of [8, 9] is that at the linearized level the coupling is the completely symmetrized trace. We briefly review this coupling below and verify that it is consistent with space-time diffeomorphism invariance. Imposing this constraint means that our results are compatible with the linear coupling of the metric to D-branes in superstring theory. If we would be interested in the coupling of D-branes in bosonic string theory to a curved background, we should change this sixth requirement to the appropriate form of the linearized coupling found there [6].

2.2.1 The linearized coupling

In [8, 9] Taylor and van Raamsdonk put forward a proposal for the coupling of the linearized graviton to the action for N free D0-branes. Moreover, the stress tensor proposed is conserved on shell [18]. This, as for the single D0-brane, must mean that the variation of the action under an infinitesimal local transformation can be cancelled by requiring that the gauge field coupling to the conserved current varies into the derivative of the parameter. Therefore, the D0-brane action of [8] with the first order coupling of the stress-tensor to the linearized graviton,

$$S = -\frac{T_0}{2}\int d\tau \mathrm{Tr}\left(\dot{X}^i\dot{X}_i\right) + \int d\tau \sum_{n=0}^{\infty} T^{ij(k_1\cdots k_n)}\partial_{k_1}\cdots\partial_{k_n}h_{ij}(0)\,, \tag{2.14}$$

where

$$T^{ij(k_1\cdots k_n)} = \frac{T_0}{2n!}\mathrm{Str}\left(\dot{X}^i\dot{X}^j X^{k_1}\cdots X^{k_n}\right), \tag{2.15}$$

must in fact be invariant under linearized infinitesimal diffeomorphisms. As has become standard, we use here the symbol **Str** for the symmetrized trace: the trace over the completely symmetrized ordering of the matrices.

Indeed, it is not so hard to check, assuming the moments of h_{ij} transform in the standard way, that the action is invariant under such linearized diffeomorphisms

$$\begin{aligned} \delta h_{ij}(x) &= \partial_{(i}\xi_{j)}(x) \quad \to \quad \delta\partial_{k_1}\cdots\partial_{k_n}h_{ij}(0) = \partial_{k_1}\cdots\partial_{k_n}\partial_{(i}\xi_{j)}(0) \\ \delta X^i &= \sum_{n=0}^{\infty}\frac{1}{n!}X^{k_1}\cdots X^{k_n}\partial_{k_1}\cdots\partial_{k_n}\xi^i(0) = \xi^i(X)\,. \end{aligned} \tag{2.16}$$

Thus, at the linearized level space-time diffeomorphisms are realized as certain "matrix-valued" diffeomorphisms. Note that there is no "unexpected" ordering in the transformation for X^i. A priori, this is not the only possible answer. The n-th derivative of the parameter ξ can only be contracted with the totally symmetric combination, but one could have internal contractions in addition. To agree with the $N \to 1$ limit such terms must involve commutators.

In the above we have ignored the potential term proportional to $[X,X]^2$ as well as the fermions required by supersymmetry. It is just as easy to show, that the action including these terms, is also invariant under the above transformations. Of course, when we include fermions we should change the metric coupling to the vielbein. For completeness, the action

$$\begin{aligned} S &= \frac{T_0}{2}\int d\tau \mathrm{Tr}\left(\dot{X}^i\dot{X}^i + \frac{1}{2}[X^i,X^j][X^i,X^j] + i\Theta\dot{\Theta} - \Theta\Gamma_i[X^i,\Theta]\right) + \\ &\quad + \sum_{n=0}^{\infty}\int d\tau T_a^{i(k_1\cdots k_n)}\partial_{k_1}\cdots\partial_{k_n}e_i^a\,, \\ T_a^{i(k_1\ldots k_n)} &= -\frac{T_0}{2n!}\mathrm{Str}\left[\left(2\dot{X}^i\dot{X}^j e_{ja}^{(0)} - 2[X^i,X^p][X^p,X^j]e_{ja}^{(0)} - \Theta\Gamma_a[X^i,\Theta]\right)X^{k_1}\cdots X^{k_n}\right], \end{aligned} \tag{2.17}$$

where $e_{ja}^{(0)} = \delta_{ja}$ is the zeroth order vielbein, is invariant under

$$\begin{aligned} \delta X^i &= \sum_{n=0}^{\infty}\frac{1}{n!}X^{k_1}\cdots X^{k_n}\partial_{k_1}\cdots\partial_{k_n}\xi^i(0) = \xi^i(X)\,, \\ \delta e_i^a(x) &= -\partial_i\xi_a(x) \qquad \Leftrightarrow \\ \delta\partial_{k_1}\cdots\partial_{k_n}e_i^a(0) &= -\partial_{k_1}\cdots\partial_{k_n}\partial_i\xi_a(0)\,. \end{aligned} \tag{2.18}$$

As we said above, we will require that the linearized coupling of the metric is as in (2.14).

For a large part we will suppress the potential and the fermions in the remainder of the paper and return to them in section 5.3.

Note that there is a different way to write (2.14), namely

$$S = -\frac{T_0}{2}\int dt \int dk \int_0^1 d\lambda\, g_{mn}(k)\mathrm{Tr}(e^{i\lambda k_j X^j}\dot{X}^m e^{i(1-\lambda)k_\ell X^\ell}\dot{X}^n)\,, \tag{2.19}$$

where $g_{mn}(k)$ is the Fourier transform of the metric. This is the form that is obtained from world-sheet calculations [6], and is reminiscent of the open Wilson lines that appear in the couplings of closed string fields to non-commutative gauge theories.

In summary, starting with a sigma model of dN^2 degrees of freedom, we want to find a d-dimensional (gauged) nonlinear sigma model built from $N \times N$ matrix-valued fields that is invariant under target space diffeomorphisms (a). This action must in addition

- (b) consist of a single group trace,
- (c) reduce to N-copies of the standard particle action if we take the matrices X to be diagonal,
- (d) yield masses proportional to the geodesic distance for off-diagonal fluctuations,
- (e) and reduce to the action (2.14) found by Taylor and van Raamsdonk, at the linearized level in the metric.

Requirements (b)–(d) are part of the axioms of D-geometry; (a) is new, and (e) is "experimental input".

3. Structure of matrix-valued diffeomorphisms

As is expressed in (2.13), given two metrics g, g' that are related by a space-time diffeomorphism, there should exist a field redefinition of the $N \times N$ matrix-coordinates $X'(X)$ such that the action remains invariant. These field redefinitions implement space-time diffeomorphism invariance, in terms of certain specific matrix-valued coordinate changes.

Unlike regular diffeomorphisms the specific redefinitions $X'(X)$ that implement space-time diffeomorphism invariance are not completely arbitrary, but it is not a priori clear what $X'(X)$ should be. It definitely should be compatible with the single trace structure, and therefore X' should be a function of X without explicit traces, e.g. a power series. Naively, one would expect that $X'(X)$ would be a suitable ordered form of $x'(x)$. We now show that there cannot be such a unique ordering, and as a consequence that the group of matrix-valued diffeomorphisms does not contain the group of space-time diffeomorphisms as a subgroup.

To explain this, let us assume for the moment that we have solved the ordering problem and consider a change of coordinates $x \to x' \to x''$, with corresponding metrics g, g', g''. Then there must be field redefinitions $X'(X)$, $X''(X')$ such that

$$S[g, X] = S[g', X'(X)] = S[g'', X''(X'(X))]\,. \tag{3.1}$$

On the other hand, by the group property, we can also consider $x \to x''$ directly, leading to $S[g, X] = S[g'', X''(X)]$. Combining with the previous equation (3.1) we find that

$$S[g'', X''(X'(X))] = S[g'', X''(X)]\,. \tag{3.2}$$

Now the only obvious invariance of the action S is the $\mathrm{U}(N)$ gauge invariance, so it hard to believe that (3.2) can be true unless

$$X''(X'(X)) = UX''(X)U^\dagger \tag{3.3}$$

with $U \in \mathrm{U}(N)$. (In other words, the metric $G_{i\alpha\beta,j\gamma\delta}$ has in general a $\mathrm{U}(N)$ group worth of Killing vectors, but no more). This means the following. Let us denote by **diff** the regular group of diffeomorphisms and by **DIFF** the group of diffeomorphisms of the $N \times N$ matrix-valued coordinates X, compatible with the single trace structure. Eq. (3.3) shows that in order for the above scenario to work there should be a projective representation of **diff** in **DIFF**, projective with respect to $\mathrm{U}(N)$.

It is, however, straightforward to show that such a representation does not exist. Consider two infinitesimal transformations

$$\begin{aligned} x'^i &= x^i + a^i_{pq} x^p x^q + b^i_{pqr} x^p x^q x^r + \mathcal{O}(x^4) && (3.4)\\ x''^i &= x'^i + c^i_{pq} x'^p x'^q + d^i_{pqr} x'^p x'^q x'^r + \mathcal{O}(x'^4) && (3.5)\\ &= x^i + (a^i_{pq} + c^i_{pq}) x^p x^q + (b^i_{pqr} + 2c^i_{ps} a^s_{qr} + d^i_{pqr}) x^p x^q x^r + \mathcal{O}(x'^4) \end{aligned}$$

and corresponding matrix redefinitions

$$\begin{aligned} X'^i &= X^i + F^i_{pq}(a) X^p X^q + G^i_{pqr}(a,b) X^p X^q X^r + \mathcal{O}(X^4) && (3.6)\\ X''^i &= X'^i + F^i_{pq}(c) X'^p X'^q + G^i_{pqr}(c,d) X'^p X'^q X'^r + \mathcal{O}(X'^4)\,, && (3.7) \end{aligned}$$

where a, b, c, d are symmetric in the lower indices but F, G are not necessarily symmetric. In order to work order by order, the function F can be a function of a (or c) only, but G can be a function of both a, b (or c, d), etc. The functions F and G tell us how to order the terms in order to lift a diffeomorphism from **diff** to **DIFF**. By inserting (3.7) into (3.6) we obtain

$$\begin{aligned} X''^i(X'(X)) &= X^i + \left(F^i_{pq}(a) + F^i_{pq}(c)\right) X^p X^q + \\ &\quad + \left(G^i_{pqr}(a,b) + G^i_{pqr}(c,d) + F^i_{ps}(c) F^s_{qr}(a) + F^i_{sr}(c) F^s_{pq}(a)\right) X^p X^q X^r\,. \end{aligned} \tag{3.8}$$

This should equal (schematically)

$$X''^i(X) = U(X^i + F^i_{pq}(a+c) X^p X^q + G^i_{pqr}(a+c, b+d+2ca) X^p X^q X^r + \mathcal{O}(X^4))U^\dagger\,. \tag{3.9}$$

To lowest order we see that $U = 1$. A generic ansatz for F and G is

$$\begin{aligned} F^i_{pq}(a) &= a^i_{pq} && (3.10)\\ G^i_{pqr}(a,b) &= b^i_{pqr} + \lambda_1 a^i_{ps} a^s_{qr} + \lambda_2 a^i_{qs} a^s_{rp} + (-\lambda_1 - \lambda_2) a^i_{rs} a^s_{pq}\,. && (3.11) \end{aligned}$$

If we insert this in (3.8) and (3.9) and demand that the equations are identical, we obtain

$$\begin{aligned}
& c^i_{ps}a^s_{qr} + c^i_{sr}a^s_{pq} + \lambda_1 a^i_{ps}a^s_{qr} + \lambda_2 a^i_{qs}a^s_{rp} + (-\lambda_1 - \lambda_2)a^i_{rs}a^s_{pq} + \\
& + \lambda_1 c^i_{ps}c^s_{qr} + \lambda_2 c^i_{qs}c^s_{rp} + (-\lambda_1 - \lambda_2)c^i_{rs}c^s_{pq} = \\
& \qquad = \lambda_1 (a+c)^i_{ps}(a+c)^s_{qr} + \lambda_2 (a+c)^i_{qs}(a+c)^s_{rp} + (-\lambda_1 - \lambda_2)(a+c)^i_{rs}(a+c)^s_{pq} + \\
& \qquad\quad + \frac{2}{3}(c^i_{ps}a^s_{qr} + c^i_{sr}a^s_{pq} + c^i_{sq}a^s_{rp})\,.
\end{aligned} \tag{3.12}$$

We immediately see that there is no solution for λ_1, λ_2 such that this equation holds. It is possible to make a more general argument that does not rely on a specific form of F and G, but we omit that here for simplicity.

The fact that **diff** is not a subgroup[1] of **DIFF** shows that in the equation (2.13),

$$S[g,X] = S[g'(g), X'(X)]\,, \tag{3.13}$$

the relation between the matrix-valued diffeomorphism $X'(X)$ and the ordinary diffeomorphism that relates g and g' is not straightforward.

A different way to reach the same conclusion is to start with the linearized form of $S[g,X]$ given in (2.14). If we require a symmetry of the form (3.13) we can use the Noether procedure to deduce the higher order couplings to the metric. Given an action consisting of a field ϕ and a gauge field A, the assumption that it is invariant under some specific variations $\delta\phi$ and δA requires the variation should vanish order by order in ϵ, where ϵ counts powers of the variation parameter and the gauge field,

$$\begin{aligned}
\delta\phi &= \epsilon\delta^{(1)}\phi + \epsilon^2\delta^{(2)}\phi + \cdots \\
\delta A &= \delta^{(0)}A + \epsilon\delta^{(1)}A + \epsilon^2\delta^{(2)}A + \cdots \\
S &= S^{(0)} + \epsilon^1 S^{(1)} + \epsilon^2 S^{(2)} + \cdots\,; \qquad \to \frac{\partial}{\partial A}S^{(0)} = 0
\end{aligned}$$

$$\begin{aligned}
\delta S = 0 = \epsilon &\left(\delta^{(1)}\phi\frac{\partial}{\partial\phi}S^{(0)} + \delta^{(0)}A\frac{\partial}{\partial A}S^{(1)}\right) + \\
&+ \epsilon^2\left(\delta^{(2)}\phi\frac{\partial}{\partial\phi}S^{(0)} + \delta^{(1)}\phi\frac{\partial}{\partial\phi}S^{(1)} + \delta^{(0)}A\frac{\partial}{\partial A}S^{(2)} + \delta^{(1)}A\frac{\partial}{\partial A}S^{(1)}\right) + \cdots\,.
\end{aligned} \tag{3.14}$$

The linear term in ϵ is just the coupling of the gauge field to the conserved current. The higher order terms are familiar from Yang-Mills theory; there $\delta^{(2)}\phi = 0$ vanishes and $\delta^{(1)}A$ together with $S^{(2)}$ completes all derivatives to covariant ones. There is no need to go to higher order in this case. For the particle action in section 2, one has to go to all orders, but the net effect is just to complete the derivative in the variation of h_{ij} to a covariant one.

If we apply the Noether method to the N D0-brane action, we find after a long and only partially illuminating calculation that one must guess higher order terms *both* in the variation of the fields and the action. These higher order terms should be of the form

$$\begin{aligned}
\delta^{(2)}X &\sim \partial h\xi[[X,X],X] \\
S^{(2)} &\sim h_{mn}h_{ij}[X,\dot{X}]^2\,,
\end{aligned} \tag{3.15}$$

where $S^{(2)}$ is symmetric in $(mn) \leftrightarrow (ij)$.

[1]Notice that **diff** is however a quotient of **DIFF**, namely by the group of matrix-valued diffeomorphisms that are of commutator type, which are the diffeomorphisms that reduce to the identity when restricted to diagonal matrices.

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Since the extra variation of X depends explicitly on the metric, this explains why it is not possible to lift **diff** to **DIFF**. In fact, the Noether procedure shows something even stronger, it shows that eq. (3.13) cannot be achieved unless we allow an explicit dependence on the metric in $X'(X)$. In other words, we will need to replace eq. (2.13) by the requirement

$$S[g,X] = S[g'(g), X'(g,X)]\,. \tag{3.16}$$

In order to have the right point particle limit, the g dependence in $X'(g,X)$ should be only through commutator terms and drop out when X is taken diagonal. The fact that diffeomorphisms depend explicitly on the background metric is reminiscent of the Seiberg-Witten map in non-commutative gauge theory [19] where the gauge parameter of the commutative theory depends non-trivially on the non-commutative gauge field. Here the matrix-valued gauge transformation depends non-trivially on the gauge field for space-time gauge transformations, which is the space-time metric.

We will take (3.16) as our basic requirement of diffeomorphism invariance. In the next section, we will try to put it in more manageable form, and study its solutions.

4. Matrix-valued diffeomorphisms from open-strings

Eq. (3.16) shows that for diffeomorphism invariance for multiple D-branes, we need to introduce a new type of symmetry. It is not a priori clear what this symmetry is, and how we can use it the determine the form of the effective action. This effective action $S[g,X]$ for multiple D-branes coupled to a background metric, can, to lowest order in the string coupling, in principle be derived by considering open string disc amplitudes with arbitrarily many open string vertex operators attached to the boundary and arbitrarily many graviton vertex operators inserted in the interior. As string theory is diffeomorphism invariant, the action derived in perturbation theory will automatically satisfy (3.16). We can thus study properties of the transformations (3.16) from string perturbation theory, even though the symmetry (3.16) is not manifest.

Amplitudes with one graviton vertex operator have been computed in [6] and yield the symmetrized trace prescription previously found by [8, 9]. Eq. (3.15) shows that new terms that are not symmetrically ordered appear once we consider two graviton vertex operators. We will not attempt to compute this or any other amplitude here, but we will only examine their general structure. We will then use this information to constrain the form of the action and these constraints will allow us to find one which obeys eq. (3.16).

The effective action only follows indirectly from string-amplitudes and one may wonder why we do not use beta-functions to reconstruct the action. It is important to realize that standard beta-function methods do not work, because we cannot turn on an arbitrary background for the matrix-valued transversal scalar fields.[2] That is because the off-diagonal elements of the transversal matrix-valued scalar fields correspond to open strings stretched between D-branes and these are generically massive. Although one can in principle try

[2] An attempt to perform such a calculation was made in [20], but the answer found there is completely symmetrically ordered, and as we already saw, this is not consistent with diffeomorphism invariance.

to directly compute effective actions for massive string degrees of freedom, see e.g. [21], it is not clear how to do that in the present context. We therefore have to resort to the calculation of correlation functions.

This raises in consequence the crucial question whether a local well-defined action for the transversal matrix-valued fields actually exists, given that the only quantity unambiguously available from string theory is the S-matrix. If we believe that there is a local formulation of string field theory, the action for N D-branes can be computed by integrating out massive string degrees of freedom. One expects a local action as long as the masses of open strings stretched between the branes is much smaller than the masses of all other massive open string degrees of freedom. This implies that the expectation value of the scalar fields (except for the U(1) part) has to be much smaller than l_s. In addition, the momenta of these fields (along the brane) have to be much smaller than $1/l_s$ so that we can neglect higher derivative terms, and string loop effects are suppressed as long as the string coupling $g_s \ll 1$. Our results will be applicable in this regime only. Indeed, provided the graviton momenta are chosen to scale in an appropriate way, it appears possible to find a consistent $\alpha' \to 0$ limit for multiple graviton scattering, just as one is able to do for a single graviton or other closed string modes [5, 6, 11, 12, 22].

So, what general properties of the effective action can we deduce from string perturbation theory, without computing any amplitudes? The correlation functions that are relevant are correlation functions of vertex operators for the transversal scalar fields and gravitons

$$V_{\mu,\alpha\beta}(k)\partial_n Y^\mu e^{ikY}(t)\,, \qquad h_{\mu\nu}(p)\partial Y^\mu \bar{\partial}\bar{Y}^\nu e^{ipY}(z,\bar{z})\,, \tag{4.1}$$

where Y^μ are the world-sheet fields, and t is a point on the boundary of the open-string worldsheet. The boundary conditions on Y^μ are that $Y^i(0) = Y^i(\pi) = \bar{x}^i$ for the Dirichlet directions $i = 1,\ldots,9$, and Neumann boundary conditions on Y^0, and we have explicitly written out the Chan-Paton indices α, β.

The effective action that one obtains from string theory is therefore of the form

$$S = S[h_{ij}(\bar{x}), \tilde{X}^i]_{\eta_{ij},\bar{x}^i}\,, \tag{4.2}$$

where η_{ij} and $\bar{x}^i$ label the background configuration, and h_{ij} and $\tilde{X}^i = (\eta + h)^{ij} V_j$ the fluctuations in the metric and transversal scalars respectively.

It is the action (4.2) that we want to relate to an action $S[g(X), X]$. That such an relation should exist follows from the fact that turning on a zero-momentum vertex operator (4.1) proportional to the identity matrix $\delta_{\alpha\beta}$, corresponds to an infinitesimal change of the boundary condition $Y^i(\sigma = 0, \pi) = \bar{x}^i$. Similarly, we can perform a finite shift of $\bar{x}^i$ by turning on a suitable condensate of the vertex operators (4.1). This implies that the $\bar{x}^i$ dependence of (4.2) can be absorbed completely in $\tilde{X}$, and the final result is an action of the form $S[g(X), X]$. This remark is crucial in our quest for a diffeomorphism invariant action (3.16). The polarization tensors $V_{i,\alpha\beta}(k)$ transform under a space-time diffeomorphism $\delta\bar{x}^i = \xi(\bar{x})^i$ as a vector. Therefore, the effective action obtained from string theory, $S[h_{ij}(\bar{x}), \tilde{X}^i]_{\eta_{ij},\bar{x}^i} = S[g(X), X]$ will be invariant under diffeomorphisms. Naturally $\eta_{ij} + h_{ij}$ transforms as a metric, and we expect $\tilde{X}^i$ to transform as a vector.

A priori though, there can be higher order corrections to the transformation rule of $\tilde{X}^i$ due to contact terms. Indeed, the argument at the beginning of this paragraph showed that $\tilde{X}$ is naturally a difference between two points, rather than a vector. However, if we employ a covariant background field expansion (see e.g. [23, 24]) to perform world-sheet calculations, all quantities that appear in the calculation will be tensors. Therefore we expect that there exists a choice of fields $\tilde{X}^i$ such that there are no higher order corrections to the transformation rule of $\tilde{X}^i$.

This appears to imply that the diffeomorphism constraint is not very restrictive; *any* action whose terms are contractions of space-time tensors constructed out of $g_{ij}(\bar{x})$ with products of vector-fields $\tilde{X}^i$ is diffeomorphism invariant. However, since the final action can be written in the form $S[g(X), X]$, there should also be symmetries of the action that shift $\bar{x}$ and act non-trivially on $\tilde{X}$, but that preserve the "coordinate" X and therefore also the action $S[g(X), X]$. The existence of such symmetries will severely constrain the action. The common setting to study these symmetries is the covariant background field expansion. We will first discuss the closely related Riemann normal coordinate system; and return to the issue of general coordinates in section 5.2.

4.1 Normal coordinates

4.1.1 Particle

As we explained above, $\bar{x}$ can be absorbed in the field $\tilde{X}$, and the final action depends only on a field X which is of the form

$$X = \bar{x} + \tilde{X} + \mathcal{O}(\tilde{X}^2)\,. \tag{4.3}$$

The higher order terms can in principle be computed using higher order correlation functions in string theory, but we will not attempt to do such a calculation here. Similar higher order terms appear in the case of a single D-brane, for which the covariant non-linear background field expansion, [23]

$$x^i = \bar{x}^i + \tilde{x}^i - \sum_{n=2}^{\infty} \frac{1}{n!}\Gamma^i_{j_1 \cdots j_n}(\bar{x})\tilde{x}^{j_1}\cdots\tilde{x}^{j_n}\,, \tag{4.4}$$

guarantees that $\tilde{x}^i$ transforms as a vector under diffeomorphisms in the backgroundfield $\delta\bar{x}^i = \xi^i(\bar{x})$. The pseudo-tensors $\Gamma^i_{j_1 \ldots j_n}(\bar{x})$ equal $\Gamma^i_{j_1 \ldots j_n}(\bar{x}) \equiv \bar{\nabla}_{(j_1} \cdots \bar{\nabla}_{j_{n-2}}\Gamma^i_{j_{n-1} j_n)}(\bar{x})$ where $\bar{\nabla}_{j_1}$ acts only on the lower indices. The action $S_{\text{particle}}[g_{ij}(x(\bar{x},\tilde{x})), x^i(\bar{x},\tilde{x})]$ expressed in terms of $\bar{x}^i$ and $\tilde{x}^i$ will contain only proper tensors evaluated at $\bar{x}^i$ contracted with the vectors $\tilde{x}^i$, $\dot{\bar{x}}$. For an easy algorithm to generate this expansion, see [24]. The proper tensors must be built from the metric or derivatives thereof, so that (suppressing time derivatives):

$$S_{\text{particle}}[g_{ij}(x(\bar{x},\tilde{x})), x^i(\bar{x},\tilde{x})] = S[g_{ij}(\bar{x}), D_{i_1}\cdots D_{i_n}R_{jklm}(\bar{x}), \tilde{x}^i]\,. \tag{4.5}$$

Again, one might have expected the full non-abelian action to be some ordered version of (4.5), where $\tilde{x}$ is replaced by $\tilde{X}$. If we assume that $\tilde{X}$ transforms as a vector, then under space-time diffeomorphisms the action obtained by ordering (4.5) will be diffeomorphism invariant for *any* ordering.

Normal coordinates is the special coordinate system where all the non-linear terms in (4.4) vanish. In this coordinate system covariant derivatives at $\bar{x}^i$ thus equal regular derivatives. Therefore, in normal coordinates the action (4.2) will have an expansion in proper space-time tensors, similar to (4.5). As (4.4) is obtained from a perturbative solution to the geodesic equation/field equation

$$\ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = 0 \tag{4.6}$$

this shows that in normal coordinates, geodesics through the origin (the point $\bar{x}$) are straight lines. Equivalently, in normal coordinates $x^i(\tau) = \tau y^i$ is a solution to the field equation.

Vice versa, the action in normal coordinates can be used to obtain the covariant action [23]. So given the action in normal coordinates, we can easily construct the action in any coordinate system. In the next section, we will discuss the implications of diffeomorphism invariance for the action in normal coordinates, but first we discuss in some more detail the case when X is matrix-valued.

4.1.2 Matrices

The action for matrix-valued fields is a special case of the dN^2-dimensional point particle action

$$S \sim \int d\tau G_{IJ}\dot{X}^I\dot{X}^J \tag{4.7}$$

introduced earlier, where I denotes the set of indices $i\alpha\beta$.

Any action of type (4.7) yields a geodesic equation and we can use this to perform a covariant background field expansion for (4.7), exactly as we did above for a point particle in d dimensions. In particular, we will specialize to

$$X = \bar{x}\mathbb{1} + \tilde{X} + \cdots . \tag{4.8}$$

Substituting this expansion back in the action (4.7) we will encounter tensors evaluated at $\bar{x}\mathbb{1}$, with indices $I, J, K, \ldots$. They are tensors under d-dimensional *matrix* diffeomorphisms of the background coordinate $\bar{x}^i$. The group of matrix diffeomorphisms, evaluated on a diagonal matrix, gives back the group of ordinary space-time diffeomorphisms. Even more is true: take a matrix-valued diffeomorphism compatible with the single trace structure, $X^I \to F(X)^I$, then

$$\left.\frac{\partial F(X)^{i\alpha\beta}}{\partial X^{j\gamma\delta}}\right|_{X=\bar{x}\mathbb{1}} = \left(\frac{\partial F(\bar{x})^i}{\partial \bar{x}^j}\right)\delta^\alpha_\gamma\delta^\beta_\delta\,. \tag{4.9}$$

We now define matrix normal coordinates as that choice of coordinates where (4.8) is linear. Due to the dN^2-dimensional origin it is clear that matrix normal coordinates are equivalent to $X^i(\tau) = \tau Y^i$ being a solution to the field equation. Furthermore, in matrix normal coordinates the action contains only dN^2-dimensional proper tensors, namely the metric, the Riemann tensor and its covariant derivatives, and is of the form (4.5) with dN^2-dimensional indices $I, J, K, \ldots$. We already argued that these dN^2-dimensional tensors are constructed out of d-dimensional space-time tensors in section 2. Equation (4.9) shows that this is also a sufficient condition for the matrix-valued action to be invariant under matrix-valued diffeomorphisms of the background field. It is the diffeomorphisms that are not purely diffeomorphisms of the background field that will significantly constrain the action expanded in space-time tensors, and it is these diffeomorphisms that we discuss next.

4.2 Diffeomorphism in normal coordinates: base point independence

We saw that the choice of normal coordinates was quite convenient, since all resulting structures can be expressed conventional space-time tensors. To understand the consequences of diffeomorphism invariance in normal coordinates, we reconsider the case of the point particle.

If the action (in arbitrary coordinates) is truly general coordinate invariant, then we could have also chosen to expand around a second set of normal coordinates $\tilde{z}^i$ centered around a *different* point $\bar{z}$. The final action $S[g_{ij}(\bar{z}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{z}), \tilde{z}^i]$ should be *identical* in form to $S[g_{ij}(\bar{x}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{x}), \tilde{x}^i]$ under the formal replacement $\bar{x} \leftrightarrow \bar{z}$, $\tilde{x} \leftrightarrow \tilde{z}$.

General coordinate invariance also means that the action is invariant if we perform the coordinate transformation from the normal coordinate system $\tilde{x}^i$ around $\bar{x}^i$ to the normal coordinate system $\tilde{z}^i$ around $\bar{z}^i$.

To obtain from an arbitrary coordinate system — including another system of normal coordinates — a system of Riemann normal coordinates around the base point $\bar{z}$, we consider geodesics of the form

$$x^i(\tau) = \bar{z}^i + \tau \tilde{z}^i + \mathcal{O}(\tau^2)\,, \tag{4.10}$$

so that $x^i(0) = \bar{z}^i$ and $\dot{x}^i(0) = \tilde{z}^i$. The higher order terms are uniquely fixed by the geodesic equation

$$\ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = 0\,. \tag{4.11}$$

By definition, $x^i(\tau = 1)$ is the expression for x^i in terms of Riemann normal coordinates $\tilde{z}^i$ around $\bar{z}^i$. In other words, we flow with unit time along a geodesic that starts at $\bar{z}^i$ and has initial velocity $\tilde{z}^i$.

Specifically we could consider the second normal coordinate system $\tilde{z}$ to be centered infinitesimally close to $\tilde{x}$: $\bar{z}^i - \bar{x}^i = \epsilon^i \ll 1$. Now since all parameters in the action for $\tilde{x}^i$ are proper tensors at $\bar{x}^i$ the total effect of the coordinate transformation is that they are just parallel transported by an infinitesimal amount ϵ^i. Explicitly this means that all terms of order ϵ or higher should cancel order by order in

$$\begin{aligned}
&S[g_{ij}(\bar{x}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{x}), \tilde{x}^i] = \\
&= S[g_{ij}(\bar{x}(\bar{z}+\epsilon)), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{x}(\bar{z}+\epsilon)), \tilde{x}^i(\tilde{z}+\epsilon(\tilde{z})+\mathcal{O}(\epsilon^2))] \\
&= S[g_{ij}(\bar{z}) + \epsilon^k D_k g_{ij}(\bar{z}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{z}) + \epsilon^k D_k D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{z}), \tilde{z}^i + \epsilon(\tilde{z}); \mathcal{O}(\epsilon^2)] \\
&\equiv S[g_{ij}(\bar{z}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{z}), \tilde{z}^i]\,.
\end{aligned} \tag{4.12}$$

An identical statement is true for $\tilde{X}^i$ and $\tilde{Z}^i$ matrix-valued. In that case, the relevant tensors are tensors in the dN^2 dimensional space described by the matrix-valued coordinates. In other words, all indices should be replaced by multi-indices $I \equiv i;\alpha\beta$. Now it is crucial that in the matrix-valued case all tensors in the dN^2 dimensional space have to be constructed out of ordinary space-time tensors. Once we express the tensors in the large dN^2 dimensional space by expressions in terms of ordinary space-time tensors, the above statement of base-point independence becomes a highly non-trivial constraint, as we will see.

To summarize, string theory gives us an action for multiple D0 branes that depends on choice of base point $\bar{x}^i$, a matrix-valued field $\tilde{X}^i$, and space-time tensors evaluated at $\bar{x}^i$. This is achieved by using Riemann normal coordinates in space-time. The action comes with a fixed ordering, which is independent of $\bar{x}$. Without loss of generality, we can always redefine $\tilde{X}^i$ so that $\tilde{X}^i$ form a set of matrix Riemann normal coordinates. (this is equivalent to requiring that $\tilde{X}^i = \tau Y^i$ is a solution of the field equations for all constant Y^i). In that case, the resulting action should be base-point independent. This means that there should exist field redefinitions of $\tilde{X}^i \to \epsilon^i \mathbb{1} + \tilde{X}^i + \mathcal{O}(\tilde{X}^2)$ such that after the field redefinition the action looks identical, including the ordering, except that the base point $\bar{x}$ has been replaced $\bar{x} + \epsilon$. The latter requirement follows naturally from string theory: by condensing transversal scalar fields one should be able to move a brane around in space-time. World-sheet calculations for both the original and final brane configuration are completely equivalent, and should therefore give identical actions, including the ordering.

4.3 Solving the constraints directly in the action

Based on the discussion above, we will discuss two ways to impose matrix normal coordinates and to solve the diffeomorphism constraint. In the next section, we do this in terms of higher dN^2 dimensional geometry, here directly using the action.

We first write down the most general action $S[g_{ij}(\bar{x}), D_{i_1} \cdots D_{i_n} R_{jklm}(\bar{x}), \tilde{X}^i]$ with arbitrary coefficients for all possible orderings of the matrices $\tilde{X}^i$. There are always two matrices carrying a proper time derivative, $d/d\tau$, and the location of these can differ depending on the ordering. The coefficients are proper tensors evaluated at $\bar{x}$ and contracted in all possible ways with a specific ordering of the matrices. Dropping the tilde on the $\tilde{X}^i$ we have

$$S = -\int d\tau \frac{1}{2} \mathrm{Tr}(\dot{X}^i \dot{X}_i) + \sum_{p,n} A^{(n)}_{i_1 \cdots i_p} \mathrm{Tr}(\dot{X}^{i_1} X^{i_2} \cdots X^{i_{n-1}} \dot{X}^{i_n} X^{i_{n+1}} \cdots X^{i_p}) \,. \tag{4.13}$$

From this action we derive the field equation

$$\begin{aligned}\frac{\delta S}{\delta X^j} = \ddot{X}^j - &\sum_{p,n,m\neq 1,n} A^{(n)}_{i_1 \cdots i_{m-1} j i_{m+1} \cdots i_p} (X^{i_{m+1}} \cdots X^{i_{n-1}} \dot{X}^{i_n} X^{i_{n+1}} \cdots X^{i_p} \dot{X}^{i_1} \cdots X^{i_{m-1}}) + \\ &+ \sum_{p,n} A^{(n)}_{j i_2 \cdots i_p} \frac{d}{d\tau} (X^{i_2} \cdots X^{i_{n-1}} \dot{X}^{i_n} X^{i_{n+1}} \cdots X^{i_p}) + \\ &+ \sum_{p,n} A^{(n)}_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_p} \frac{d}{d\tau} (X^{i_{n+1}} \cdots X^{i_p} \dot{X}^{i_1} X^{i_2} \cdots X^{i_{n-1}}) \,. \end{aligned} \tag{4.14}$$

The first constraint is the requirement that the action S we started with in eq. (4.13) is in normal coordinates around $\bar{x}$. This is equivalent to requiring that $X^i(\tau) = \tau Y^i$ is a solution of the field equation (4.14). Setting $\tau = 1$ this imposes a constraint on the to be determined coefficients $A^{(n)}$.

$$0 = \sum_{p,n,m\neq 1,n} A^{(n)}_{i_1\cdots i_{m-1} j i_{m+1}\cdots i_p}(Y^{i_{m+1}}\cdots Y^{i_{n-1}}Y^{i_n}Y^{i_{n+1}}\cdots Y^{i_p}Y^{i_1}\cdots Y^{i_{m-1}}) - \\ - \sum_{p,n}(p-2)A^{(n)}_{j i_2\cdots i_p}(Y^{i_2}\cdots Y^{i_{n-1}}Y^{i_n}Y^{i_{n+1}}\cdots Y^{i_p}) - \\ - \sum_{p,n}(p-2)A^{(n)}_{i_1\cdots i_{n-1} j i_{n+1}\cdots i_p}(Y^{i_{n+1}}\cdots Y^{i_p}Y^{i_1}Y^{i_2}\cdots Y^{i_{n-1}})\,. \tag{4.15}$$

Next we find the coordinate transformation to a nearby system of normal coordinates Z^i around $\bar{z} = \bar{x} + \epsilon$. Substituting $\dot{X}^i(0) = Z^i$ in the matrix-geodesic connecting X^i with ϵ^i

$$X^i(\tau) = \epsilon^i + \tau\dot{X}^i(0) + \frac{\tau^2}{2!}\ddot{X}^i(0) + \cdots \tag{4.16}$$

and solving for higher derivatives of X^i in terms of Z^i and ϵ^i using the field equation, we find the coordinate transformation between X^i and Z^i

$$X^i = \epsilon^i + Z^i + \sum_p \Delta^i_{j_1\cdots j_p}(\epsilon, A^{(n)})Z^{j_1}\cdots Z^{j_p}\,. \tag{4.17}$$

Note that the imposition of the first constraint, that $X^i = \tau Y^i$ is a solution to the field equation, means that in $\Delta^i_{j_1\cdots j_p}$ the zeroth order term in ϵ^i vanishes. Substituting (4.17) in the action, we keep the term linear in ϵ, and denote it with $\delta S(\epsilon, Z)$. This change in the action should be identical to a shift in the base point $\bar{x}^i \to \bar{z}^i = \bar{x}^i + \epsilon^i$. In normal coordinates this is achieved by parallel transporting all tensors $A^{(n)}$ by an infinitesimal amount ϵ. Thus the condition of base point independence is equivalent to the requirement that

$$\delta S(\epsilon, Z) = -\int d\tau \sum_{p,n}\epsilon^k D_k A^{(n)}_{i_1\cdots i_p}\mathrm{Tr}(\dot{X}^{i_1}X^{i_2}\cdots X^{i_{n-1}}\dot{X}^{i_n}X^{i_{n+1}}\cdots X^{i_p})\,. \tag{4.18}$$

In appendix A.1 we work out the details of this procedure to order X^4.

In section 2 we saw that the linearized coupling found by Taylor and van Raamsdonk [8, 9]

$$S = -\frac{1}{2}\int \mathrm{Tr}\dot{X}^i\dot{X}_i + \frac{1}{2}\int\frac{1}{n!}\mathrm{STr}\left(\dot{X}^i\dot{X}^jX^{k_1}\cdots X^{k_n}\right)\partial_{k_1}\cdots\partial_{k_n}h_{ij}\,, \tag{4.19}$$

is diffeomorphism invariant to first order. It should therefore satisfy the condition of base-point independence, and it is instructive to verify this. In terms of (matrix) Riemann normal coordinates, the linearized coupling equals

$$\partial_{k_1}\cdots\partial_{k_n}h_{ij} = \frac{\alpha_n}{n!}D_{(k_1}\cdots D_{k_{n-2}}R_{k_{n-1}|(ij)|k_n)} + \mathcal{O}(\mathrm{Riem}^2)\,, \tag{4.20}$$

where $\alpha_n = (n-1)/(n+1)$. In appendix A.2 we show that this action is consistent with base-point independence. As one should expect, one can in fact derive explicit value of

α_n this way. At higher orders the above method becomes cumbersome and the matrix geometry approach is more efficient.

With the use of the full Riemann tensor and symmetrized covariant derivatives thereof, it may seem as if we have obtained more information than the linearized results of Taylor and van Raamsdonk. This is of course not true, as the above result only holds in the normal coordinate frame, where the linearized expression for the metric at $\partial \cdots \partial h(\bar{x})$ in (4.20) exactly equals the symmetrized covariant derivatives of the Riemann tensor.

4.4 Solving the constraints in matrix geometry

The action we start with is an action of the form

$$S = \frac{1}{2} \int d\tau g_{IJ}(X) \dot{X}^I \dot{X}^J \,, \tag{4.21}$$

where I, J are dN^2 dimensional indices that are of the form $i; \alpha\beta$ with $i = 1, \ldots, d$ etc. What we want is to express $G_{IJ}(X)$ in terms of $g_{ij}(x)$. Our strategy is to first go to Riemann normal coordinates, in which the metric has an expansion of the form

$$\begin{aligned} g_{IJ}(X) = \delta_{IJ} + \frac{1}{3} R_{IKLJ}(\bar{x}) X^K X^L + \frac{1}{6} \nabla_M R_{IKLJ}(\bar{x}) X^K X^L X^M + \frac{2}{45} R_{IKL\lambda}(\bar{x}) \times \\ \times R^\lambda{}_{MNJ}(\bar{x}) X^K X^L X^M X^N + \frac{1}{20} R_{IKLJ;MN}(\bar{x}) X^K X^L X^M X^N + O(X^5) \,. \end{aligned} \tag{4.22}$$

Expressing $g_{IJ}(X)$ in terms of $g_{ij}(x)$, now amounts to expressing all curvature tensors and their covariant derivatives $R_{IJKL}(\bar{x}), \nabla_M R_{IJKL}(\bar{x})$, etc. in terms of $R_{ijkl}(\bar{x}), \nabla_m R_{ijkl}(\bar{x})$, etc. In other words, we will write down general expressions for R_{IJKL} and its derivatives in terms of R_{ijkl} and its derivatives times tensors containing all the Chan-Paton indices. For instance, we could try

$$R_{IJKL} = R_{ijkl} \delta_{\alpha_2\beta_1} \delta_{\beta_2\gamma_1} \delta_{\gamma_2\delta_1} \delta_{\delta_2\alpha_1} + \cdots \,, \tag{4.23}$$

where $I = i; \alpha_1\alpha_2$, $J = j; \beta_1\beta_2$, etc.

We impose the following constraints on the tensors $R_{IJKL}(R_{ijkl}, \ldots)$ and their derivatives.

(*a*) The curvature tensor R_{IJKL} has to have to usual symmetries, meaning antisymmetric in the first and second pair of indices, and symmetric under exchange of the first and second pair.

(*b*) It should have cyclic symmetry.

(*c*) Covariant derivatives ∇_I should obey the Bianchi identity.

(*d*) Multiple covariant derivatives should, when anti-symmetrized, yield the usual rules, e.g. $[\nabla_I, \nabla_J] R_{KLMN} = R_{IJK}{}^P R_{PLMN} + 3$ more terms.

(*e*) The U(N) indices should be contracted in a single trace style, as the action contains a single trace.

(f) To have the right behavior under change of basepoint, we require that

$$\delta^{\alpha\beta}\nabla_{i\alpha\beta}(\text{anything}) = \nabla_i(\text{anything})\,. \tag{4.24}$$

(g) To first order in the curvature, the symmetrized trace prescription, found in [8, 9] should emerge.

(h) It should have the right U(1) limit for diagonal matrices.

Conditions (a) through (d) guarantee that the curvature tensors and their derivatives all come from a single metric $g_{IJ}(X)$. If we would just write down some random tensors that would violate one of the conditions (a) through (d), they could never correspond to the curvature tensors of some metric $g_{IJ}(X)$.

Condition (e) implies that the curvatures and its covariant derivatives can be written as sums of ordinary tensors with indices $i, j, k, l, \ldots$ times tensors $\Delta_{\alpha\beta\gamma\delta}$, where the latter are defined as

$$\Delta_{\alpha\beta\gamma\delta} = \delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_1}\delta_{\gamma_2\delta_1}\delta_{\delta_2\alpha_1} \tag{4.25}$$

and similarly with more indices. The tensors Δ are cyclically invariant, and yield single trace expressions when contracted with matrix-valued coordinates.

Condition (f) is crucial, and expresses the constraint of base-point independence. In the way we have set things up, it is guaranteed that under a change of base point all tensors transform as $T \to \epsilon^i \delta^{\alpha_1\alpha_2}\nabla_{i\alpha_1\alpha_2}T$. However, we want that this is the same as changing the base point in all ordinary tensors that appear in T, without affecting the matrix structure. Therefore, we demand that $\epsilon^i\delta^{\alpha_1\alpha_2}\nabla_{i\alpha_1\alpha_2}T = \epsilon^i\nabla_i T$ for all T.

Finally, conditions (g) and (h) are obvious.

These are the "axioms" of the matrix geometry. Once the curvature tensors have been specified, the action in normal coordinates follows directly by applying the expansion (4.22) to (4.21).

We have solved these equations for terms up to order X^6, for which we need the curvature tensor and its first and second derivatives.

There are 120 linearly independent terms containing two $\dot{X}$'s, four X's, and a product of two Riemann tensors with one index contracted, that one can write down. We found that conditions (a) through (h) leave 32 of those undetermined. The full result is very lengthy, and for illustrational purposes we give a two-parameter family of solutions in appendix B. We postpone a discussion of this solution to section 5, and conclude here by showing in some more detail how one determines R_{IJKL} in terms of R_{ijkl}.

4.5 The matrix curvature tensor

The most general form of the curvature that will give a single trace answer is

$$\begin{aligned} R_{i\alpha_1\alpha_2 j\beta_1\beta_2 k\gamma_1\gamma_2 l\delta_1\delta_2} = {} & T^1_{ijkl}\Delta_{\alpha\beta\gamma\delta} + T^2_{ijkl}\Delta_{\alpha\beta\delta\gamma} + T^3_{ijkl}\Delta_{\alpha\gamma\beta\delta} + \\ & + T^4_{ijkl}\Delta_{\alpha\gamma\delta\beta} + T^5_{ijkl}\Delta_{\alpha\delta\beta\gamma} + T^6_{ijkl}\Delta_{\alpha\delta\gamma\beta} \end{aligned} \tag{4.26}$$

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with Δ defined in (4.25). The symmetries of the curvature tensor reduce this to

$$\begin{aligned} R_{i\alpha_1\alpha_2 j\beta_1\beta_2 k\gamma_1\gamma_2 l\delta_1\delta_2} = & \; T^1_{ijkl}\Delta_{\alpha\beta\gamma\delta} - T^1_{ijlk}\Delta_{\alpha\beta\delta\gamma} + T^3_{ijkl}\Delta_{\alpha\gamma\beta\delta} - \\ & - T^1_{jikl}\Delta_{\alpha\gamma\delta\beta} + T^3_{klij}\Delta_{\alpha\delta\beta\gamma} + T^1_{jilk}\Delta_{\alpha\delta\gamma\beta}\,, \end{aligned} \tag{4.27}$$

where $T^1_{ijkl} = T^1_{klij}$, and T^3 is anti-symmetric in the first and second pair of indices. Cyclic symmetry then implies

$$\begin{aligned} T^1_{ijkl} - T^1_{lijk} + T^3_{ljik} &= 0 \\ -T^1_{ijlk} + T^3_{iljk} + T^1_{kijl} &= 0\,. \end{aligned} \tag{4.28}$$

Any of these two identities can be used to write

$$T^3_{ijkl} = T^1_{ikjl} - T^1_{kjli}\,, \tag{4.29}$$

where one can use the pair symmetry of T^1. Thus, everything is expressed in terms of a single tensor T^1 that satisfies

$$\begin{aligned} T^1_{ijkl} - T^1_{klij} &= 0 \\ T^1_{ikjl} - T^1_{kjli} + T^1_{jkil} - T^1_{kilj} &= 0\,, \end{aligned} \tag{4.30}$$

where the second identity follows from the antisymmetry of T^3. If T^1 can be expressed in terms of R_{ijkl}, it must be of the form

$$T^1_{ijkl} = \lambda R_{ijkl} + \mu R_{ikjl}\,. \tag{4.31}$$

Both (4.30) and (4.30) are satisfied for all λ and μ. Thus symmetry dictates that the curvature is of the form

$$\begin{aligned} R_{i\alpha_1\alpha_2 j\beta_1\beta_2 k\gamma_1\gamma_2 l\delta_1\delta_2} = & \; (\lambda R_{ijkl} + \mu R_{ikjl})\Delta_{\alpha\beta\gamma\delta} + (\lambda R_{ijkl} + \mu R_{ilkj})\Delta_{\alpha\beta\delta\gamma} + \\ & + (\lambda R_{ijkl} + 2\mu R_{ijkl})\Delta_{\alpha\gamma\beta\delta} + (\lambda R_{ijkl} + \mu R_{ilkj})\Delta_{\alpha\gamma\delta\beta} + \\ & + (\lambda R_{ijkl} + 2\mu R_{ijkl})\Delta_{\alpha\delta\beta\gamma} + (\lambda R_{ijkl} + \mu R_{ikjl})\Delta_{\alpha\delta\gamma\beta}\,. \end{aligned} \tag{4.32}$$

It is quite curious that a two-parameter family of curvatures exists at this level. This can be compared to the similar result derived in appendix A.1 where we implemented diffeomorphism invariance directly in the action. In the U(1) limit we get $6(\lambda+\mu)R_{ijkl}$, and therefore

$$6(\lambda + \mu) = 1\,. \tag{4.33}$$

To see what term one would get in the action, we contract with

$$\dot{X}^{i\alpha_1\alpha_2} X^{j\beta_1\beta_2} \dot{X}^{k\gamma_1\gamma_2} X^{l\delta_1\delta_2} \tag{4.34}$$

to obtain

$$\begin{aligned} \delta S = & \; (\lambda R_{ijkl} + \mu R_{ikjl})\mathrm{Tr}(\dot{X}^i X^j \dot{X}^k X^l) + \\ & + (\lambda R_{ijkl} + \mu R_{ilkj})\mathrm{Tr}(\dot{X}^i X^j X^l \dot{X}^k) + \\ & + (\lambda R_{ijkl} + 2\mu R_{ijkl})\mathrm{Tr}(\dot{X}^i \dot{X}^k X^j X^l) + \\ & + (\lambda R_{ijkl} + \mu R_{ilkj})\mathrm{Tr}(\dot{X}^i \dot{X}^k X^l X^j) + \\ & + (\lambda R_{ijkl} + 2\mu R_{ijkl})\mathrm{Tr}(\dot{X}^i X^l X^j \dot{X}^k) + \\ & + (\lambda R_{ijkl} + \mu R_{ikjl})\mathrm{Tr}(\dot{X}^i X^l \dot{X}^k X^j)\,. \end{aligned} \tag{4.35}$$

We see that the μ terms do not contribute to the $\mathrm{Tr}(\dot{X}X\dot{X}X)$ structure but do contribute to the $\mathrm{Tr}(\dot{X}\dot{X}XX)$ structure. Thus, in order to get the symmetrized answer we need to take $\mu = 0$.

To conclude, symmetries restrict the form to (4.32). If we in addition require the right form of the action, i.e. completely symmetrized to leading order in R, we need to take $\lambda = 1/6$ and $\mu = 0$, i.e.

$$R_{i\alpha_1\alpha_2 j\beta_1\beta_2 k\gamma_1\gamma_2 l\delta_1\delta_2} = R_{ijkl}\Delta_{\mathrm{Sym}(\alpha\beta\gamma\delta)}\,. \tag{4.36}$$

The Taylor-van Raamsdonk answer for the linearized coupling is therefore not a consequence of diffeomorphism invariance alone, but needs to be imposed by hand. This is confirmed by world-sheet calculations: in the case of the bosonic string it has been shown in [6] that the linearized coupling to the metric is not completely symmetrically ordered, but it should still be diffeomorphism invariant.

A completely symmetrized answer is still a solution for ∇R,

$$\nabla_{m\epsilon_1\epsilon_2} R_{i\alpha_1\alpha_2 j\beta_1\beta_2 k\gamma_1\gamma_2 l\delta_1\delta_2} = \nabla_m R_{ijkl}\Delta_{\mathrm{Sym}(\epsilon\alpha\beta\gamma\delta)}\,. \tag{4.37}$$

This structure breaks down for $\nabla\nabla R$, as one can easily check that the completely symmetrized answer does not give the right answer for $[\nabla,\nabla]R$ (point (d) in the criteria above). Therefore, the action for N D0 branes cannot be the fully symmetrized answer of the action for a single D0 brane. To get an idea of the complicated nature of the action we give in appendix B the action for a special choice of parameters up to order X^6.

5. Properties of the solution

The action that we constructed explicitly up to order X^6, and which is given in appendix B, is quite ugly. Even with 30 of the 32 free parameters removed, one is still left with a long expression. The result does exhibit properties of the general action that are worth observing.

One property is the fact that the action contains new contractions beyond the ones that are present in the U(1) limit. For instance, in the U(1) limit the R^2 terms are always such that each $\dot{x}$ is contracted with a separate curvature tensor. This is no longer the case in the non-abelian action, which always contains R^2 terms where two $\dot{X}$'s are contracted with a single curvature tensor. This is clear from the action given in appendix B, and one can verify that it remains true for arbitrary values of the 32 free parameters. Thus, the non-abelian action is not just a naive ordering of the abelian one, but new terms are needed as well.

Once we extend the analysis to higher order, at least some of the 32 free parameters will be fixed. For instance it is not hard to show that the coefficient of the term in the action proportional to $\Delta S \sim R_{ijkl}R^i_{mnp}\mathrm{Tr}(\mathrm{Asym}(\dot{X}^j X^k X^l)\mathrm{Asym}(\dot{X}^m X^n X^p))$ will only be determined by enforcing base-point independence at the next order. Ideally, all of them would be fixed in this way, but this appears unlikely. It is known that the actions for D branes in the bosonic string and the superstring are different once a non-trivial metric

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is introduced [6]; on the other hand both should be covariant. Therefore, there exists more than one covariant non-abelian kinetic term. It is quite likely that many covariant kinetic terms exist, that all differ by commutator terms. It would be quite useful to have a classification of all covariant kinetic terms, as this would provide valuable insight in the structure of D-geometry.

5.1 Reality of the action

One issue we have not yet discussed at all is the reality of the action. The fields X are hermitean, and reality of the action given in appendix B will impose certain reality conditions on the various free parameters. For example, reality of a combination $a\mathrm{Tr}(X^iX^jX^k) + b\mathrm{Tr}(X^kX^jX^i)$ requires that a and b are each others complex conjugate. Once we determine the reality conditions, one can in principle remove all terms from the action that have imaginary coefficients in a self-consistent way. It is possible that this will reduce the number of free parameters somewhat, but we have not examined this in detail.

5.2 Nonnormal coordinates

Our analysis made crucial use of normal coordinates. We could also have chosen to work in a completely arbitrary coordinate system. In that case, the resulting action should be an expansion in $\tilde{X}^i$ whose coefficients are arbitrary multi-index objects built out of the metric and its derivatives. Diffeomorphism invariance implies that if we compute the action in two different coordinate systems, and use the same regularization scheme and method of computation for each, the resulting actions should look exactly identical, including their ordering. The only difference is in the explicit form of the metric that appears. Since the actions computed this way depend on arbitrary metrics and do not rely on normal coordinates it may appear that covariance of these actions can impose additional constraints. This is not true, however. Each action can always be written in terms of matrix Riemann normal coordinates. Since the actions are identical, except for a change of metric, the changes of coordinates will also be identical except for the same change of metric. In Riemann normal coordinates, we will therefore again end up with identical actions including the ordering, except for the metric that appears. This is precisely the structure that we worked with and we see that it contains the same amount of information.

Conversely, suppose that we want to write down the action for N D0 branes in arbitrary coordinates. To accomplish that, we first go to space-time normal coordinates and construct the action as we explained. Next, we undo the change to space-time normal coordinates. This requires that we find a matrix generalization of the map from space-time normal coordinates to the coordinate system we started out with. We can take any matrix generalization that reduces in the U(1) limit to the required change of coordinates. Different choices simply give actions that are related by a field redefinition and therefore yield equivalent physics. If we agree on a specific ordering, e.g. completely symmetrized, and use that for any change of normal coordinates to the original coordinates, the resulting action in the original coordinates will always look identical, includ-

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ing the ordering. The only difference will be in the metric that appears. Thus, we can construct an action for N D0 branes in arbitrary coordinate systems that has all the properties that we would expect from string theory considerations. Different choices of orderings correspond to field redefinitions $\delta X \sim [X,X]$ which involve commutators and vanish for diagonal matrices. These are, of course, outside the scope of space-time diffeomorphisms.

5.3 The potential, fermions and other contributions

In this paper, we have so far focused exclusively on the kinetic terms in the action. Of course, the complete non-abelian DBI action has many more terms; it has a potential term, fermionic terms, Wess-Zumino terms and higher order derivative terms. Obviously, these terms all have to be separately covariant. For those terms that only involve the transversal scalar fields X, the notion of covariance is identical to that for the kinetic terms. In other words, the terms should admit an expansion in terms of space-time tensors multiplying expression in $\tilde{X}$, and they should be base-point independent. Suppose that the kinetic term is given up to a certain order, then we know the transformations for $\tilde{X}$ that implement a change of base-point up to that order. The same transformation should also yield a change of base-point for all other terms in the action, which are thereby severely constrained.

For example, it is possible to write down a potential in curved space in terms of the matrix valued metric $G_{IJ}(X)$ that is completely covariant, and that reduces to the usual potential in flat space. To write the potential, we introduce a matrix version of the vielbein by writing[3]

$$G_{IJ}(X) = \sum_A E^A_{I^t} E^A_{J^t}\,, \qquad G^{IJ} E^A_{I^t} E^B_{J^t} = \delta^{AB}\,, \tag{5.1}$$

where $A,B = 1 \cdots dN^2$. Here I^t denotes $i\beta\alpha$ for $I = i\alpha\beta$. The base point independence of the action $G_{IJ}\dot{X}^I\dot{X}^J$ implies a simple transformation rule for $E^A_{I^t}\dot{X}^I$ under a change of basepoint, namely it rotates by an $\mathrm{SO}(dN^2)$ transformation. Notice that $E^A_I \equiv E^A_{i\alpha\beta}$ is a matrix, which we will denote by E^A_i. Thus it is $\mathrm{Tr}(E^A_i \dot{X}^i)$ that transforms nicely. Now observe that taking the time derivative acts as a derivation on the algebra of matrices (i.e. it satisfies the Leibnitz rule), but so does the operation $V : X \to [X,V]$ for fixed V. Therefore, $\mathrm{Tr}(E^A_i[X^i,V]) = \mathrm{Tr}([E^A_i,X^i]V)$ will also transform nicely under a change of base-point, if we keep V inert. We claim that a covariant version of the potential is

$$\frac{1}{2}\mathrm{Tr}([X^i,X^j]^2) \to \sum_{A,B}\frac{1}{4}\mathrm{Tr}([E^A_i,X^i][E^B_j,X^j][E^A_k,X^k][E^B_l,X^l])\,. \tag{5.2}$$

We just explained that $\mathrm{Tr}[E^A_i,X^i]$ transforms under a base-point changing transformation in a simply way, via an $\mathrm{SO}(dN^2)$ transformation. Eq. (5.2) is therefore automatically

[3] Notice that these vielbeins do not have simple $\mathrm{U}(N)$ transformations rules. One can also introduce twisted vielbeins via $G_{i\alpha\beta,j\gamma\delta} = E^r_{i\alpha\delta}E^r_{j\gamma\beta}$, and which do transform in the adjoint representation under $\mathrm{U}(N)$. It is not clear whether these twisted vielbeins play any role in matrix geometry.

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covariant. One can also verify that in flat space, it reduces to the usual answer.[4] Moreover, we know the full linearized coupling of the potential term to the metric [8, 9, 6]; it is given by a fully symmetrized expression. A somewhat tedious calculation shows that this is correctly reproduced by (5.2). This is strong evidence that eq. (5.2) is the curved-space version of the potential. One can try to use a similar strategy to study the Ramond-Ramond couplings in the action.

The use of vielbeins for the potential is encouraging, as its introduction is a prerequisite for the inclusion of fermions. Their correct treatment will demand a separate analysis. We will get a structure where fermions and fields $\tilde{X}$ multiply space-time tensors. However, the transformations that shift the base-point may have to be modified. In the U(1) limit, such modifications are absent, but we do not know any reason why they should be absent in the U(N) limit as well.

This also raises the question how world-volume supersymmetry and κ-symmetry could be connected to our notion of covariance. Both supersymmetry and κ-symmetry impose conditions on the background fields, so we can no longer consider the most general coupling. In particular, κ-symmetry typically requires the background fields to be on-shell. In such a situation the constraint of covariance becomes less stringent. On the other hand, to combine either supersymmetry or kappa-symmetry with the non-abelian structure will significantly constrain the possible actions that we can write down. This idea was pursued for supersymmetry in [1, 7] and for κ-symmetry in [15].

5.4 The spectrum of quadratic fluctuations

According the the axioms of D-geometry of Douglas, the spectrum of quadratic fluctuations around a diagonal matrix should agree with the geodesic lengths between the space-time points appearing in the diagonal matrices. This criterion is natural if we identify those diagonal entries with the locations of the D-branes, and the off-diagonal elements with the open strings stretched between them. We will now demonstrate that with the choice of potential (5.2), the masses of the off-diagonal fluctuations are indeed *exactly* given by the geodesic distances between the separated branes. For simplicity, we will assume that the action contains terms with real coefficients only.

The masses of off-diagonal fluctuations are obtained by writing the matrices X as the sum of a diagonal matrix Δ with time-independent diagonal entries λ^i_α, plus an arbitrary off-diagonal matrix $Y^i_{\alpha\beta}(t)$. We expand the action to second order in Y and to all orders in λ^i_α, and from the result we read off the masses.

We will work in the gauge $A_0 = 0$ and enforce the Gauss law explicitly. It states that[5]

$$G_{i\alpha\beta,j\gamma\rho}\dot{X}^i_{\alpha\beta}X^j_{\rho\delta} - G_{i\alpha\beta,j\rho\delta}\dot{X}^i_{\alpha\beta}X^j_{\gamma\rho} = 0\,. \tag{5.3}$$

The expansion of the kinetic term is easy, because the λ^i_α are time-independent. We need the metric $G_{i\alpha\beta,j\gamma\delta}$ evaluated on the diagonal matrix with entries λ^i_α. This is of the form

[4] Use that $G_{i\alpha\beta,j\gamma\delta}|_{\text{flat}} = \eta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma}$. See also the previous footnote.

[5] Recall that the action equals $S = \frac{1}{2}\mathrm{Tr}(E^A_i D_0 X^i)\mathrm{Tr}(E^A_j D_0 X^j)$ (note that this does not violate the U(N) single-trace structure).

(see footnote 4)

$$G_{i\alpha\beta,j\gamma\delta}(\Delta) \equiv S_{ij\alpha\beta}(\Delta)\delta_{\alpha\delta}\delta_{\beta\gamma}\,. \tag{5.4}$$

Clearly, $S_{ij\alpha\beta}$ depends only on λ^k_α and λ^k_β. The fact that the metric is symmetric implies

$$S_{ij\alpha\beta} = S_{ji\beta\alpha}\,. \tag{5.5}$$

Moreover, we assumed that the action is real with real coefficients, which yields the additional symmetry

$$S_{ij\alpha\beta} = S_{ji\alpha\beta}\,. \tag{5.6}$$

With this definition of S, the expansion of the kinetic term is given by

$$\mathcal{L}_{\rm kin} = S_{ij\alpha\beta}\dot{Y}^i_{\alpha\beta}\dot{Y}^j_{\beta\alpha}\,. \tag{5.7}$$

Next we turn to the expansion of the potential term (5.2). Although this is not obvious from the form (5.2), the only contributions to second order in Y arise when two of the four explicit X's that appear in (5.2) are diagonal, and two of them are off-diagonal. This is most easily seen by writing (5.2) explicitly in terms of indices, and by reexpressing the vielbeins in terms of the metric using (5.1). One finds that the potential can also be expressed in terms of S and is given by

$$\mathcal{L}_{\rm pot} = (S_{ik\alpha\beta}S_{jl\alpha\beta} - S_{ij\alpha\beta}S_{kl\alpha\beta})(\lambda^i_\alpha - \lambda^i_\beta)(\lambda^j_\alpha - \lambda^j_\beta)Y^k_{\alpha\beta}Y^l_{\beta\alpha}\,. \tag{5.8}$$

At the order we are working in, the Gauss law constraint translates into

$$S_{ij\alpha\beta}(\lambda^i_\alpha - \lambda^i_\beta)\dot{Y}^j_{\alpha\beta} = 0 \qquad \rightarrow \qquad S_{ij\alpha\beta}(\lambda^i_\alpha - \lambda^i_\beta)Y^j_{\alpha\beta} = \text{const.} \tag{5.9}$$

This decouples one of the nine transverse scalars. Because we want to expand around a solution, we put the constant in (5.9) equal to zero, which causes the first term in eq. (5.8) to vanish.

It is then obvious from (5.7) and (5.8) that the masses of the eight remaining transversal scalars are all identical and equal to

$$M^2_{\alpha\beta} = S_{ij\alpha\beta}(\Delta)(\lambda^i_\alpha - \lambda^i_\beta)(\lambda^j_\alpha - \lambda^j_\beta)\,. \tag{5.10}$$

It is not yet clear that this expression equals the geodesic distance squared. It is true that the geodesic distance between two points can be written in the form (5.10) where $S_{ij\alpha\beta}$ has a good power series expansion in $\lambda^i_\alpha, \lambda^i_\beta$. To prove that (5.10) is identical to the geodesic distance squared, the notion of covariance we introduced in this paper is crucial. Because of this covariance, we can without loss of generality put one of the two points, say λ^i_β, equal to zero, and at the same time we can use Riemann normal coordinates around zero. The geodesic distance in Riemann normal coordinates between 0 and λ^i_α is simply given by

$$d^2(\lambda_\alpha, 0) = g_{ij}(0)\lambda^i_\alpha\lambda^j_\alpha\,, \tag{5.11}$$

so we are done if we can show that

$$S_{ij\alpha\beta}(\Delta)\lambda^i_\alpha\lambda^j_\alpha = g_{ij}(0)\lambda^i_\alpha\lambda^j_\alpha \tag{5.12}$$

in Riemann normal coordinates. From the action we already know that $S_{ij\alpha\beta} = g_{ij}(0) + \mathcal{O}(\lambda_\alpha^2)$, and all higher order terms in S are contractions of tensors built out of the Riemann tensor with the λ_α^i. A simple counting argument shows that in $S_{ij\alpha\beta}(\Delta)\lambda_\alpha^i\lambda_\alpha^j$ every term, except the leading one proportional to $g_{ij}(0)$, contains at least one Riemann tensor with at least three indices contracted with λ_α^i. This vanishes by the Bianchi identity, and we conclude that (5.12) is indeed true.

Because of the covariance and base-point independence of the metric, the result that the masses of off-diagonal fluctuations equals the geodesic length will remain valid in any coordinate system. This leads to the interesting conclusion that the geodesic distance in space-time can be directly read off from $G_{IJ}(X)$, evaluated on diagonal X. There is no need to integrate a line element along a geodesic. More precisely, if we write the kinetic term in the form

$$\sum_t \mathrm{Tr}(P^t_{ij}(X)\dot{X}^i Q^t_{ji}(X)\dot{X}^j) \tag{5.13}$$

with some set P^t_{ij}, Q^t_{ji}, then the geodesic distance $d(x,y)$ satisfies

$$d(x,y)^2 = \sum_t P^t_{ij}(x)Q^t_{ji}(y)(x-y)^i(x-y)^j\,. \tag{5.14}$$

As a final comment, we observe the the expansion of the potential to second order also immediately implies that vacuum manifold of the potential consists of M^N/S_N, i.e. the moduli space of N unordered points in M. This was one of the axioms of D-geometry as put forward in [1].

5.5 Connection with non-commutative geometry

The fact that for multiple D-branes, the transverse coordinates are replaced by matrices, suggests a role for non-commutative geometry in the action for multiple D-branes. In non-commutative geometry, the space of functions is replaced by a non-commutative algebra, and the obvious candidate here would be to consider the algebra

$$\mathcal{A} = C^\infty(M) \otimes M_N(C)\,. \tag{5.15}$$

This algebra does not yet carry any metric information. From the representation theoretical point of view, it is very close to the original algebra $C^\infty(M)$ (they are Morita equivalent). Following Connes, the construction of a riemannian structure requires a spectral triple $(\mathcal{A},\mathcal{H},D)$ which in addition to $\mathcal{A}$ also contains a Hilbert space $\mathcal{H}$ and a self-adjoint operator D obeying certain properties [25]. It would be interesting to find triples $(\mathcal{A},\mathcal{H},D)$ that describe in a natural way metrics relevant for multiple D-branes, and that incorporate the notion of covariance. The form of the action and the potential suggest that the vielbein E_I^A introduced in (5.1) will play an important role in such a construction.

Another possible connection with non-commutative geometry is with the non-commutative geometry description of D-branes in the presence of a B-field in space-time (or a magnetic field strength on the brane) [3, 26]. The point is that the structure of an abelian non-commutative gauge theory is very similar to that of a non-abelian commutative gauge theory. In both cases, fields no longer commute, and the field strengths are non-linear. Moreover, non-commutative gauge theories can be constructed starting from a non-abelian commutative theory by expanding around suitable backgrounds and taking $N \to \infty$ (see e.g. [27] and references therein). The latter connection shows it should be possible to relate the coupling of non-commutative gauge theories to gravity to the coupling of non-abelian D-brane actions to gravity. This was indeed the approach taken in [12] where the stress-tensor of non-commutative gauge theories was derived in this way. It would be very interesting to see whether a similar connection can be made at higher order. The following remarks suggestive that such connections may exist.

1 There is a close relation between the world-sheet calculations involving gravitons for non-commutative gauge theories and D-branes [11, 6]. Both lead to a result where certain operators are smeared along some kind of Wilson line. This structure becomes more complicated when more than one graviton vertex operator is involved,[6] and the precise geometry underlying such calculations has not been uncovered.

2 Non-commutative spaces are constructed in deformation quantization from commutative spaces equipped with a closed two-form, and this is also how they arise in string theory. The diffeomorphisms of the commutative space that preserve the two-form become gauge transformations of the non-commutative space (they are just canonical transformations). Thus, there should also be a relation between the coupling of a gauge field in non-commutative gauge theory and the coupling of the graviton in non abelian gauge theory.

3 There is a one-to-one correspondence between single trace expressions one can write down in terms of matrix valued coordinates X^i, and expressions involving ordinary coordinates and a closed two-form B^{IJ}. Open string amplitudes depend only on the combination $\mathcal{F} = B + F$ and T-duality maps $F \leftrightarrow [X, X]$. This map extends to

$$\begin{aligned} X^i X^j + X^j X^i &\leftrightarrow 2x^i x^j \\ [X^i, X^j] &\leftrightarrow B^{ij} \\ [X^k, [X^i, X^j]] &\leftrightarrow B^{kl}\partial_l B^{ij} \end{aligned} \tag{5.16}$$

etcetera.[7] This suggests there should be a relation (some kind of Seiberg-Witten map) between a single D brane in a transversal B-field, and multiple D-branes without a transversal B-field.

6. Discussion and conclusion

In this paper we discussed the notion of covariance for multiple D-branes in transversal curved space. We used this notion to constrain the action and studied the kinetic term

[6]We would like to thank H. Ooguri for a discussion of this point.

[7]Due to the Jacobi relation, the B-field must obey certain constraints. These are equivalent to the integrability constraints in deformation quantization [28].

explicitly up to sixth order in the fields. We also gave an explicit form of the potential in terms of the kinetic term, and showed that the resulting action satisfies all axioms of D-geometry.

We have uncovered a glimpse of an intricate geometrical structure that encodes the behavior of multiple D-branes in curved space. The precise mathematical structure underlying this geometry, and the corresponding stringy fuzziness of space-time, are still waiting to be uncovered.

Nevertheless, there are several directions in which the results here can be extended. A covariant formulation of the coupling to other closed string background fields, and an investigation of analogs of the Myers effect in curved backgrounds are two such issues. We would also like to know the behavior of D-branes in black hole backgrounds, where we expect the deviations from classical geometry to become particularly relevant.

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A. Diffeomorphism invariance in the action: explicit calculations

In this appendix we give some explicit details of calculations where we impose normal coordinates and diffeomorphism invariance directly in the action. In section A.1 we construct the action up to order X^4, and in section A.2 we give the linearized coupling to the graviton to all orders.

A.1 A calculation to order X^4

We see from (4.18) that the constraint at order p in Z^i requires one to start at order $p+1$ in the action (4.13). The first non-trivial constraint will arise at order three. The most general action with arbitrary coefficients for all inequivalent orderings of the matrices X^i to order four is

$$\begin{aligned} S = -T_0 \int d\tau \frac{1}{2}\mathrm{Tr}(\dot{X}^i \dot{X}_i) + A^{(3)}_{i_1 i_2 i_3} \mathrm{Tr}(\dot{X}^{i_1} \dot{X}^{i_2} X^{i_3}) + \\ + A^{(4);(1)}_{i_1 i_2 i_3 i_4} \mathrm{Tr}(\dot{X}^{i_1} \dot{X}^{i_2} X^{i_3} X^{i_4}) + A^{(4);(2)}_{i_1 i_2 i_3 i_4} \mathrm{Tr}(\dot{X}^{i_1} X^{i_2} \dot{X}^{i_3} X^{i_4}) + \cdots . \end{aligned} \tag{A.1}$$

Up to a factor of $-T_0$, this action has as field equation

$$\begin{aligned} \frac{\delta}{\delta X^j} S = -\ddot{X}_j - A^{(3)}_{j i_2 i_3} \frac{d}{d\tau}(\dot{X}^{i_2} X^{i_3}) - A^{(3)}_{i_1 j i_3} \frac{d}{d\tau}(X^{i_3} \dot{X}^{i_1}) + A^{(3)}_{i_1 i_2 j} \dot{X}^{i_1} \dot{X}^{i_2} - \\ - A^{(4);(1)}_{j i_2 i_3 i_4} \frac{d}{d\tau}(\dot{X}^{i_2} X^{i_3} X^{i_4}) - A^{(4);(1)}_{i_1 j i_3 i_4} \frac{d}{d\tau}(X^{i_3} X^{i_4} \dot{X}^{i_1}) + \end{aligned}$$

$$
\begin{aligned}
&+ A^{(4);(1)}_{i_1i_2i_3i_4}(X^{i_4}\dot{X}^{i_1}\dot{X}^{i_2}\delta_j^{i_3} + \dot{X}^{i_1}\dot{X}^{i_2}X^{i_3}\delta_j^{i_4}) - A^{(4);(2)}_{ji_2i_3i_4}\frac{d}{d\tau}(X^{i_2}\dot{X}^{i_3}X^{i_4}) - \\
&- A^{(4);(2)}_{i_1i_2ji_4}\frac{d}{d\tau}(X^{i_4}\dot{X}^{i_1}X^{i_2}) + A^{(4);(2)}_{i_1i_2i_3i_4}(\dot{X}^{i_3}X^{i_4}\dot{X}^{i_1}\delta_j^{i_2} + \dot{X}^{i_1}X^{i_2}\dot{X}^{i_3}\delta_j^{i_4}) \,. \qquad \text{(A.2)}
\end{aligned}
$$

The condition that S is in fact in normal coordinates requires that each of the combinations

$$
\begin{aligned}
0 = &\left[A^{(3)}_{ji_2i_3} + A^{(3)}_{i_3ji_2} - A^{(3)}_{i_2i_3j}\right] Y^{i_2}Y^{i_3} + \\
&+ \Big[2A^{(4);(1)}_{ji_2i_3i_4} + 2A^{(4);(1)}_{i_4ji_2i_3} - A^{(4);(1)}_{i_3i_4ji_2} - A^{(4);(1)}_{i_2i_3i_4j} + \\
&\quad + 2A^{(4);(2)}_{ji_2i_3i_4} + 2A^{(4);(2)}_{i_3i_4ji_2} - A^{(4);(2)}_{i_4ji_2i_3} - A^{(4);(2)}_{i_2i_3i_4j}\Big] Y^{i_2}Y^{i_3}Y^{i_4} \qquad \text{(A.3)}
\end{aligned}
$$

vanish.

Next we establish the form of the coordinate transformation to normal coordinates Z^i around the nearby basepoint ϵ^i. This is given by

$$
X^i(\tau) = X^i(0) + \tau\dot{X}^i(0) + \frac{\tau^2}{2!}\ddot{X}^i(0) + \cdots \qquad \text{(A.4)}
$$

for $\tau = 1$ after substituting $X^i = \epsilon^i \cdot \mathbb{1}$ and $\dot{X}^i(0) = Z^i$. To find an expression for $\ddot{X}^i(0)$ we use the field equation. We find to order ϵ and order two in Z^i (recall that the zeroth order term in ϵ vanishes due to (A.3))

$$
\begin{aligned}
\ddot{X}_j(0) = &-\Big[A^{(4);(1)}_{ji_2i_3i_4} + A^{(4);(1)}_{ji_2i_4i_3} + A^{(4);(1)}_{i_3ji_2i_4} + A^{(4);(1)}_{i_3ji_4i_2} - A^{(4);(1)}_{i_2i_3ji_4} - A^{(4);(1)}_{i_2i_3i_4j} + \\
&\quad + A^{(4);(2)}_{ji_2i_3i_4} + A^{(4);(2)}_{ji_4i_2i_3} + A^{(4);(2)}_{i_3i_4ji_2} + A^{(4);(2)}_{i_2i_3ji_4} - A^{(4);(2)}_{i_3ji_2i_4} - A^{(4);(2)}_{i_2i_4i_3j}\Big] Z^{i_2}Z^{i_3}\epsilon^{i_4} + \\
&+ \mathcal{O}(\epsilon^2) \,. \qquad \text{(A.5)}
\end{aligned}
$$

Something special happens in the next step, when we substitute the coordinate change (A.4) into the action, but this is peculiar to this order. It does not occur at higher orders. Making the substitution we find that at order Z^3 all terms involving the coefficients $A^{(4);(1)}_{i_1i_2i_3i_4}$ and $A^{(4);(2)}_{i_1i_2i_3i_4}$ cancel and the linear term in ϵ in the action is just

$$
\begin{aligned}
\delta S|_\epsilon &= \int d\tau \delta A^{(3)}_{i_1i_2i_3}\mathrm{Tr}(\dot{Z}^{i_1}\dot{Z}^{i_2}X^{i_3}) \\
&= \int d\tau \delta\epsilon^k D_k A^{(3)}_{i_1i_2i_3}\mathrm{Tr}(\dot{Z}^{i_1}\dot{Z}^{i_2}X^{i_3}) \,. \qquad \text{(A.6)}
\end{aligned}
$$

The requirement of general coordinate invariance is thus that the tensor $A^{(3)}_{i_1i_2i_3}$ is covariantly constant. There is, however, no proper tensor with three indices that one can construct from the metric or derivatives thereof.[8] Hence $A^{(3)}_{i_1i_2i_3}$ in fact vanishes identically. At this order general coordinate invariance imposes no constraint.

[8]This is not quite true, e.g. $D^kR_{ki_2i_3i_4}$, but consistency with the U(1) limit allows only the use of a single derivative on the metric.

We are thus left solely with the two constraints (A.3). The first one is again trivially satisfied as there does not exist a proper tensor with three indices. There exists one proper tensor built from two derivatives of the metric and with four indices: the Riemann tensor. The coefficients $A^{(4);(1)}_{i_1 i_2 i_3 i_4}$ and $A^{(4);(2)}_{i_1 i_2 i_3 i_4}$ are thus proportional to a particular combination of Riemann tensors. The various identities that the latter satisfies mean that there are only two independent combinations and therefore

$$\begin{aligned} A^{(4);(1)}_{i_1 i_2 i_3 i_4} &= a_1 R_{i_1 i_2 i_3 i_4} + a_2 R_{i_1 i_3 i_2 i_4} \\ A^{(4);(2)}_{i_1 i_2 i_3 i_4} &= b_1 R_{i_1 i_2 i_3 i_4} + b_2 R_{i_1 i_3 i_2 i_4} \,. \end{aligned} \tag{A.7}$$

Our task has reduced to determining the coefficients a_1, a_2 and b_1, b_2. Substituting (A.7) into the normal coordinate constraint (A.3) we find that the coefficients must obey

$$((-2a_1 - 2b_1) R_{i_1 i_2 i_3 j} + (a_1 + 4b_1 + 6b_2) R_{i_1 i_3 i_2 j}) Y^{i_1} Y^{i_2} Y^{i_3} = 0 \,. \tag{A.8}$$

The Riemann normal coordinate and base-point independence constraints thus leave at order three a two-parameter class of solutions. At order four and five, the system remains under-determined. To fix this final ambiguity we will require that the solution is consistent with the linearized result obtained in [8]. To analyze this, we substitute the most general form for the order-four coefficients (A.7) into the action to get

$$\begin{aligned} S^{(4)} &= a_1 R_{i_1 i_2 i_3 i_4} \mathrm{Tr}(\dot{X}^{i_1} \dot{X}^{i_2} X^{i_3} X^{i_4}) + a_2 R_{i_1 i_3 i_2 i_4} \mathrm{Tr}(\dot{X}^{i_1} \dot{X}^{i_2} X^{i_3} X^{i_4}) + \\ &\quad + b_1 R_{i_1 i_2 i_3 i_4} \mathrm{Tr}(\dot{X}^{i_1} X^{i_2} \dot{X}^{i_3} X^{i_4}) + b_2 R_{i_1 i_3 i_2 i_4} \mathrm{Tr}(\dot{X}^{i_1} X^{i_2} \dot{X}^{i_3} X^{i_4}) \,. \end{aligned} \tag{A.9}$$

At this level there is a combination of Riemann tensors which equates to a double derivative on h_{ij} at the origin in normal coordinates.[9] The relation

$$\partial_\mu \Gamma^\tau_{\nu\rho} = \frac{1}{3} R_{\mu(\nu\rho)\tau} \tag{A.10}$$

implies that

$$\partial_\mu \partial_\rho g_{\nu\tau} = -\frac{1}{3} R_{\mu(\nu\tau)\rho} \,. \tag{A.11}$$

Each independent ordering in $S^{(4)}$ must combine to a double derivative on $h_{ij} = g_{ij} - \eta_{ij}$ with both indices contracted with the *dotted* X^i's in order that we can find agreement with the linearized result. This is the case if

$$\begin{aligned} a_1 &= -\frac{\lambda}{3} \qquad & a_2 &= \frac{2\lambda}{3} \\ b_1 &= \frac{2\rho}{3} \qquad & b_2 &= -\frac{\rho}{3} \end{aligned} \tag{A.12}$$

[9] In normal coordinates derivatives of the metric at the origin are related. Using

$$\partial_{(\mu_1} \ldots \partial_{\mu_{n-2}} \Gamma^\tau{}_{\mu_{n-1}\mu_n)} = 0$$

one can show that

$$\partial_{(\mu_1} \ldots \partial_{\mu_{n-1}} g_{\mu_n)\tau} = \frac{1}{2} \partial_\tau \partial_{(\mu_1} \ldots \partial_{\mu_{n-2}} g_{\mu_{n-1}\mu_n)} \,.$$

in which case $S^{(4)}$ reduces to

$$S^{(4)} = \lambda\partial_{i_3}\partial_{i_4}g_{i_1i_2}\mathrm{Tr}(\dot{X}^{i_1}\dot{X}^{i_2}X^{i_3}X^{i_4}) + \rho\partial_{i_2}\partial_{i_4}g_{i_1i_3}\mathrm{Tr}(\dot{X}^{i_1}X^{i_2}\dot{X}^{i_3}X^{i_4}) . \tag{A.13}$$

Now if $\lambda = 2\rho$ this combines to

$$\begin{aligned} S^{(4)} &= \rho\partial_{i_3}\partial_{i_4}g_{i_1i_2}\Big(\mathrm{Tr}(\dot{X}^{i_1}\dot{X}^{i_2}X^{i_3}X^{i_4}) + \mathrm{Tr}(\dot{X}^{i_1}X^{i_3}\dot{X}^{i_2}X^{i_4}) + \mathrm{Tr}(\dot{X}^{i_1}X^{i_3}X^{i_4}\dot{X}^{i_2})\Big) \\ &= 3\rho\mathrm{Str}(\dot{X}^{i_1}\dot{X}^{i_2}X^{i_3}X^{i_4}) . \end{aligned} \tag{A.14}$$

Demanding consistency with the linearized result already yields a unique solution

$$3\rho = \frac{T_0}{2\,2!} \qquad \to \qquad \rho = \frac{T_0}{2}\frac{1}{6} . \tag{A.15}$$

We still should check whether the Matrix-normal-coordinate conditions are satisfied, i.e. do the equations (A.8) hold? Substituting the solution

$$\begin{aligned} a_1 &= -\frac{2\rho}{3} \qquad & a_2 &= \frac{4\rho}{3} \\ b_1 &= \frac{2\rho}{3} \qquad & b_2 &= -\frac{\rho}{3} \end{aligned} \tag{A.16}$$

we see that

$$2a_1 + 2b_1 = 0 \qquad a_1 + 4b_1 + 6b_2 = 0 \tag{A.17}$$

vanish as required.

Finally let us quickly check the consistency with the U(1) result, though this is implied by agreement with the linearized result. This requires that

$$a_2 + b_1 = \frac{T_0}{2}\frac{1}{3} \tag{A.18}$$

which is also seen to hold.

A.2 The linearized coupling to all orders

As preliminary, note that the symmetrized trace, **Str**, obeys the nice identity

$$\mathrm{STr}(ABC\ldots) = \mathrm{Tr}(A\,\mathrm{Sym}(BC\ldots)) \tag{A.19}$$

which may be proven using the expression

$$\mathrm{STr}(A^pB^q) = \frac{1}{(p+q)!}\mathrm{Tr}\frac{d^p}{d\alpha^p}\frac{d^q}{d\beta^q}(\alpha A + \beta B)^{p+q}|_{\alpha,\beta=0} . \tag{A.20}$$

Then the field equation which follows from the action (4.19) with the substitution (4.20) equals

$$\begin{aligned} 0 = \ddot{X}_\mu - \frac{d}{dt}\frac{\alpha_n}{n!}\mathrm{Sym}(\dot{X}^jX^{k_1}\cdots X^{k_n})R_{k_1(\mu j)k_2\cdots k_n} + \\ + \frac{n\alpha_n}{2n!}\mathrm{Sym}(\dot{X}^i\dot{X}^jX^{k_1}\cdots X^{k_{n-1}})R_{k_1(ij)k_2\cdots k_{n-1}\mu} , \end{aligned} \tag{A.21}$$

where the tensors are defined as[10]

$$R_{k_1(\mu j)k_2\cdots k_n} \equiv \frac{1}{n!} D_{(k_n} \cdots D_{k_3} R_{k1|\mu j|k_2)} + (\mu \leftrightarrow j) \tag{A.22}$$

and are thus completely symmetric in the k_i indices. Notice that we keep α_n arbitrary. The value of α_n should equal $(n-1)/(n+1)$ to have the right U(1) limit, but here we will present an alternative derivation of this value based on diffeomorphism invariance.

Field-Equation constraint. Checking that we really are in Riemann normal coordinates, we need to show that the field equation has $X^i = tY^i$ at $t=1$ as solution. This is obvious because one always ends up with a Riemann tensor symmetrically contracted over three indices.

Base-point transformation. Next we need to find the basepoint transformation

$$X^i = \epsilon^i + t\dot{X}^i + \frac{t^2}{2}\ddot{X}^i + \cdots \tag{A.23}$$

with $\dot{X}^i|_{t=0} = Z^i$. $\ddot{X}^i$ and higher derivatives are related to Z^i through the field equation (A.21). We find

$$\begin{aligned}\left(\frac{d}{dt}\right)^{m+2} X_\mu = \frac{\alpha_n}{n!}\Bigg\{ &\left(\frac{d}{dt}\right)^{m+1} \mathrm{Sym}(\dot{X}^j X^{k_1}\cdots X^{k_n}) R_{k_1(\mu j)k_2\cdots k_n} - \\ &- \frac{n}{2}\left(\frac{d}{dt}\right)^{m} \mathrm{Sym}(\dot{X}^i \dot{X}^j X^{k_1}\cdots X^{k_{n-1}}) R_{k_1(ij)k_2\cdots k_{n-1}\mu}\Bigg\}. \end{aligned} \tag{A.24}$$

From this we see that if we are only interested in terms linear in the Riemann tensor that we may ignore multiple d/dt derivatives on potential terms already involving Riemann tensors. To remain linear in ϵ we need exactly one X^i with the remainder $\dot{X}^i$. Hence $n = m+2$. Using the identity for the symmetrized product (A.19) it is then straightforward to show that

$$\begin{aligned}\left(\frac{d}{dt}\right)^{m+2} X_\mu\Bigg|_{t=0} = \alpha_{m+2}\Bigg\{ &\epsilon^{k_1}\mathrm{Sym}(Z^j Z^{k_2}\cdots Z^{k_{m+2}}) R_{k_1(\mu j)k_2\cdots k_{m+2}} - \\ &- \frac{\epsilon^{k_1}}{2}\mathrm{Sym}(Z^i Z^j Z^{k_2}\cdots Z^{k_{m+1}}) R_{k_1(ij)k_2\cdots k_{m+1}\mu}\Bigg\}. \end{aligned} \tag{A.25}$$

Thus to linear order in both ϵ^i and Riemann tensors the coordinate transformation from one set of Riemann normal coordinates to a nearby one is

$$\begin{aligned}X_\mu = \epsilon_\mu + Z_\mu + \frac{\alpha_n}{n!}\epsilon^{k_1}\Bigg\{ &\mathrm{Sym}(Z^j Z^{k_2}\cdots Z^{k_n}) R_{k_1(\mu j)k_2\cdots k_n} - \\ &- \frac{1}{2}\mathrm{Sym}(Z^i Z^j Z^{k_2}\cdots Z^{k_{n-1}}) R_{k_1(ij)k_2\cdots k_{n-1}\mu}\Bigg\}. \end{aligned} \tag{A.26}$$

[10]Parenthetical symmetrization does not include a $1/n!$, viz.

$$A_{(a}B_{b)} = A_a B_b + A_b B_a \qquad etc.$$

Thus, symmetrized expressions have weight $n!$ instead of 1.

To simplify this expression we notice that the symmetrization of the Z_i does not yet correspond with the properties of the tensor it is contracted with. Let's rewrite

$$\begin{aligned}
&\mathrm{Sym}(Z^j Z^{k_2}\cdots Z^{k_n})R_{k_1(\mu j)k_2\cdots k_n} = \\
&\quad = \mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})R_{k_1(\mu p_1)p_2\cdots p_n} \\
&\quad = \mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})\left\{\frac{2}{n}D_{p_n}\cdots D_{p_3}R_{k_1(\mu p_1)p_2} + \frac{1}{n}\sum_{\ell=4}^{n+1} D_{p_n}\cdots{}_{p_\ell k_1 p_{\ell-1}}\cdots D_{p_4}R_{p_2(\mu p_1)p_3}\right\} \\
&\quad = -\frac{1}{n}\mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})R_{p_1(k_1\mu)p_2\cdots p_n}\,. \qquad \text{(A.27)}
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathrm{Sym}(Z^i Z^j Z^{k_2}\cdots Z^{k_{n-1}})R_{k_1 ij k_2\cdots k_{n-1}\mu} = \\
&\quad = \mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})\left\{\frac{2(n-2)!}{n!}D_{p_n}\cdots D_{p_3}R_{k_1p_1p_2\mu} + \cdots\right\} \\
&\quad = \frac{1}{n(n-1)}\mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})R_{p_1(k_1\mu)p_2\cdots p_n}\,. \qquad \text{(A.28)}
\end{aligned}$$

This implies that the transformation $\Delta X_\mu = X_\mu(Z) - Z_\mu$ is simply

$$\Delta X_\mu = \epsilon_\mu - \frac{\alpha_n}{(n-1)n!}\epsilon^{k_1}\mathrm{Sym}(Z^{p_1}\cdots Z^{p_n})R_{p_1(k_1\mu)p_2\cdots p_n}\,. \qquad \text{(A.29)}$$

Invariance of the action. Now we check whether the transformation of the action under the above coordinate transformation can be cancelled by a covariant change of the coupling constants: the Riemann tensors. The change in the action equals ΔX_μ times the field equation in (A.21)

$$\begin{aligned}
\Delta S = \mathrm{Tr}\Bigg(&-\frac{d}{dt}\Delta X^\mu\left[\dot Z_\mu - \frac{\alpha_n}{n!}\mathrm{Sym}(\dot Z^\alpha Z^{k_1}\cdots Z^{k_n})R_{k_1(\mu\alpha)k_2\cdots k_n}\right] + \\
&+ \Delta X^\mu\left[\frac{n\alpha_n}{2n!}\mathrm{Sym}(\dot Z^\alpha \dot Z^\beta Z^{k_1}\cdots Z^{k_{n-1}})R_{k_1(\alpha\beta)k_2\cdots k_{n-1}\mu}\right]\Bigg)\,. \qquad \text{(A.30)}
\end{aligned}$$

It is straightforward to see that the single d/dt derivative on ΔX_μ yields

$$\frac{d}{dt}\Delta X_\mu = -\frac{n\alpha_n}{(n-1)n!}\epsilon^{k_1}\mathrm{Sym}(\dot Z^{p_1}Z^{p_2}\cdots Z^{p_n})R_{p_1(k_1\mu)p_2\cdots p_n}\,. \qquad \text{(A.31)}$$

Then to linear order in ϵ^i and Riemann tensors the change in the action is

$$\begin{aligned}
\Delta S &= \frac{n\alpha_n}{(n-1)n!}\epsilon^{k_1}\mathrm{STr}(\dot Z^\mu \dot Z^{p_1}Z^{p_2}\cdots Z^{p_n})R_{p_1(k_1\mu)p_2\cdots p_n} + \\
&\quad + \frac{n\alpha_n}{2n!}\epsilon^\mu\mathrm{STr}(\dot Z^\alpha \dot Z^\beta Z^{k_1}\cdots Z^{k_{n-1}})R_{k_1(\alpha\beta)k_2\cdots k_{n-1}\mu} \\
&= \frac{n\alpha_n}{n!}\epsilon^{k_1}\mathrm{STr}(\dot Z^\alpha \dot Z^\beta Z^{p_2}\cdots Z^{p_n})\left(\frac{1}{(n-1)}R_{\beta(k_1\alpha)p_2\cdots p_n} + \frac{1}{2}R_{p_2(\alpha\beta)p_3\cdots p_n k_1}\right)\,. \qquad \text{(A.32)}
\end{aligned}$$

At the end we expect that this reduces to

$$\Delta S \sim \epsilon^{k_1}D_{k_1}\mathrm{STr}(\dot Z^\alpha \dot Z^\beta Z^{p_1}\cdots)R_{p_1(\alpha\beta)p_2\cdots}\,. \qquad \text{(A.33)}$$

Thus we have to somehow extract a single covariant derivative from the symmetrized Riemann tensors. Substituting their definition we find that

$$\begin{aligned}&\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})R_{\beta(k_1\alpha)p_2\cdots p_n} = \\ &= \mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\left[\frac{2}{n}D_{p_n}\cdots D_{p_3}R_{\beta(k_1\alpha)p_2}+\frac{1}{n}D_{p_n}\cdots{}_\beta\cdots D_{p_4}R_{p_2(k_1\alpha)p_3}\right]\end{aligned} \tag{A.34}$$

and

$$\begin{aligned}&\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})R_{p_2(\alpha\beta)p_3\cdots p_n k_1} = \\ &= \mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\left[\frac{2}{n}D_{p_n}\cdots D_{p_3}R_{k_1(\alpha\beta)p_2}+\frac{1}{n}D_{p_n}\cdots{}_{k_1}\cdots D_{p_4}R_{p_2(\alpha\beta)p_3}\right].\end{aligned} \tag{A.35}$$

Combining these equations we find that

$$\begin{aligned}\Delta S = \frac{\alpha_n}{n!}\epsilon^{k_1}\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\Bigg(&\frac{(n-2)}{(n-1)}D_{p_n}\cdots D_{p_3}R_{k_1(\alpha\beta)p_2}+\\ &+\frac{1}{2}D_{p_n}\cdots{}_{k_1}\cdots D_{p_4}R_{p_2(\alpha\beta)p_3}+\\ &+\frac{2}{(n-1)}D_{p_n}\cdots{}_\beta\cdots D_{p_4}R_{k_1p_2p_3\alpha}\Bigg).\end{aligned} \tag{A.36}$$

Next we use the Bianchi identity to compare the three terms

$$D_{p_3}R_{k_1\alpha\beta p_2} = -D_{k_1}R_{\alpha p_3\beta p_2} - D_\alpha R_{p_3k_1\beta p_2}\,. \tag{A.37}$$

One finds

$$\begin{aligned}\Delta S = \frac{\alpha_n}{n!}\epsilon^{k_1}\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\Bigg(&\frac{2(n-2)}{(n-1)}(-D_{p_n}\cdots D_{k_1}R_{\alpha p_3\beta p_2}-D_{p_n}\cdots D_\alpha R_{p_3k_1\beta p_2})+\\ &+\frac{1}{2}D_{p_n}\cdots{}_{k_1}\cdots D_{p_4}R_{p_2(\alpha\beta)p_3}+\\ &+\frac{2}{(n-1)}D_{p_n}\cdots{}_\beta\cdots D_{p_4}R_{k_1p_2p_3\alpha}\Bigg).\end{aligned} \tag{A.38}$$

Finally, since we are only interested in terms linear in Riemann tensors we may at this stage ignore all commutators of the covariant derivatives, and we end up with solely

$$\begin{aligned}\Delta S &= \frac{\alpha_n}{n!}\epsilon^{k_1}\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\left(\frac{(n+1)(n-2)}{(n-1)}D_{k_1}D_{p_n}\cdots D_{p_4}R_{p_2\alpha\beta p_3}\right)\\ &= \frac{\alpha_n(n-2)}{2n!}\frac{(n+1)}{(n-1)}\epsilon^{k_1}\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})D_{k_1}R_{p_2(\alpha\beta)p_3\cdots p_n}\end{aligned} \tag{A.39}$$

which can be cancelled by suitable covariant derivatives of the Riemann tensors

$$\delta S = -\frac{\alpha_{n-1}}{2(n-1)!}\mathrm{STr}(\dot{Z}^\alpha\dot{Z}^\beta Z^{p_2}\cdots Z^{p_n})\epsilon^{k_1}D_{k_1}R_{p_2(\alpha\beta)p_3\cdots p_n} \tag{A.40}$$

provided

$$\begin{aligned} \frac{\alpha_{n-1}}{(n-1)!} &= \frac{\alpha_n}{n!}\frac{(n-2)(n+1)}{(n-1)} \\ \Leftrightarrow \quad \alpha_{n-1}\frac{n}{n-2} &= \alpha_n\frac{n+1}{n-1} \end{aligned} \tag{A.41}$$

with the obvious solution

$$\alpha_n = \frac{n-1}{n+1} \tag{A.42}$$

as expected from the U(1) limit.

B. The action to order X^6

Here we present a two parameter family of actions that satisfy the diffeomorphism constraint. The most general action depends on 32 free parameters, we have chosen the two parameters in such a way the action is relatively simple. Unfortunately, we have not succeeded in finding values for the parameters for which the action can be written in a nice compact form.

The action is obtained from (4.22) and is given by

$$\begin{aligned} g_{IJ}\dot{X}^I\dot{X}^J &= \dot{X}^I\dot{X}^I + \frac{1}{3}R_{KIJL}X^IX^J\dot{X}^K\dot{X}^L + \\ &+ \frac{1}{6}\nabla_M R_{KIJL}X^IX^JX^M\dot{X}^K\dot{X}^L + \\ &+ \frac{1}{20}\nabla_N\nabla_M R_{KIJL}X^IX^JX^MX^N\dot{X}^K\dot{X}^L + \\ &+ \frac{2}{45}R_{KIJP}R_{PMNL}X^IX^JX^MX^N\dot{X}^K\dot{X}^L\,. \end{aligned} \tag{B.1}$$

The non-abelian generalization for the first three terms is simply the maximally symmetrized version. The last term becomes, using the curvature tensor found in section 4,

$$\frac{2}{45}R_{kijp}R_{pmnl}\mathrm{TrSym}(X^iX^j\dot{X}^k)\mathrm{Sym}(X^mX^n\dot{X}^l)\,. \tag{B.2}$$

This term is uniquely determined and already not fully symmetrically ordered.

The main problem is the one but last term. It has one contribution which is just

$$\frac{1}{20}\nabla_n\nabla_m R_{kijl}\mathrm{TrSym}(X^iX^jX^mX^n\dot{X}^k\dot{X}^l) \tag{B.3}$$

plus many R^2 terms. The latter depend on 32 free parameters. If we put 30 of those equal to zero, in order to obtain a presentable expression, we find that the R^2 terms are of the form

$$\begin{aligned} &\mathrm{Tr}(X^nX^mX^i\dot{X}^j\dot{X}^kX^l)\times T^1_{nmiklj} + \\ &+ \mathrm{Tr}(X^nX^mX^i\dot{X}^kX^l\dot{X}^j)\times T^2_{nmiklj} + \\ &+ \mathrm{Tr}(X^nX^i\dot{X}^kX^mX^l\dot{X}^j)\times T^3_{nmiklj} \end{aligned} \tag{B.4}$$

with

$$
\begin{aligned}
T^1_{nmiklj} = & \left(-\frac{7}{120}-2\beta\right)R_{mklp}R_{nijp}+\left(-\frac{7}{360}+\beta\right)R_{mlkp}R_{nijp}+\\
&+\left(\frac{17}{180}+2\beta\right)R_{mjlp}R_{nikp}+\left(-\frac{7}{90}-\beta\right)R_{mljp}R_{nikp}+\\
&+\left(-\frac{1}{40}+\frac{\alpha}{5}+\frac{\beta}{5}\right)R_{jklp}R_{nimp}+\left(\frac{1}{10}-\frac{3\alpha}{5}+\frac{9\beta}{10}\right)R_{jlkp}R_{nimp}+\\
&+\left(\frac{67}{360}+4\beta\right)R_{mklp}R_{njip}+\left(-\frac{7}{120}-2\beta\right)R_{mlkp}R_{njip}+\\
&+\left(\frac{1}{20}+\beta\right)R_{mikp}R_{njlp}+\left(\frac{1}{45}+\beta\right)R_{mkip}R_{njlp}+\\
&+\left(-\frac{11}{60}-4\beta\right)R_{mjlp}R_{nkip}+\left(-\frac{17}{180}+2\beta\right)R_{mljp}R_{nkip}+\\
&+\left(\frac{13}{180}-\alpha+\beta\right)R_{mijp}R_{nklp}+\left(-\frac{13}{360}+\frac{2\alpha}{5}-\frac{3\beta}{5}\right)R_{mjip}R_{nklp}+\\
&+\left(-\frac{4}{45}+\beta\right)R_{mikp}R_{nljp}+\left(\frac{1}{20}+\beta\right)R_{mkip}R_{nljp}+\\
&+\left(\frac{1}{120}+4\alpha\right)R_{mijp}R_{nlkp}+\left(\frac{13}{180}-\alpha+\beta\right)R_{mjip}R_{nlkp}+\\
&+\left(-\frac{1}{40}+\frac{\alpha}{5}+\frac{\beta}{5}\right)R_{ijkp}R_{nlmp}+\left(-\frac{3}{40}+\frac{2\alpha}{5}-\frac{11\beta}{10}\right)R_{ikjp}R_{nlmp}+\\
&+\left(-\frac{7}{360}-\frac{7\alpha}{5}-\frac{7\beta}{5}\right)R_{jklp}R_{nmip}+\left(-\frac{3}{40}+\frac{9\alpha}{5}-\frac{6\beta}{5}\right)R_{jlkp}R_{nmip}+\\
&+\left(\frac{2}{45}+\frac{6\alpha}{5}+\frac{6\beta}{5}\right)R_{ijkp}R_{nmlp}+\left(-\frac{7}{360}-\frac{3\beta}{2}\right)R_{ikjp}R_{nm,l,p}
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
T^2_{nmiklj} = & \frac{1}{180}R_{mjkp}R_{nilp}+\frac{7}{180}R_{mkjp}R_{nilp}+\\
&+\left(\frac{11}{360}-\frac{6\alpha}{5}+\frac{3\beta}{10}\right)R_{jklp}R_{nimp}+\left(-\frac{11}{180}+\frac{9\alpha}{5}-\frac{6\beta}{5}\right)R_{jlkp}R_{nimp}+\\
&+\frac{5}{72}R_{milp}R_{njkp}+\frac{1}{36}R_{mlip}R_{njkp}+\\
&+\left(-\frac{1}{9}+\alpha-\beta\right)R_{mikp}R_{njlp}+\left(-\frac{19}{180}-\frac{2\alpha}{5}-\frac{12\beta}{5}\right)R_{mkip}R_{njlp}+\\
&+\left(-\frac{19}{180}-\frac{2\alpha}{5}-\frac{12\beta}{5}\right)R_{iklp}R_{njmp}+\left(\frac{29}{360}+\frac{2\alpha}{5}+\frac{7\beta}{5}\right)R_{ilkp}R_{njmp}+\\
&+\frac{1}{36}R_{milp}R_{nkjp}+\frac{1}{36}R_{mlip}R_{nkjp}+\\
&+\left(-\frac{67}{360}-4\beta\right)R_{mijp}R_{nklp}+\left(\frac{23}{180}+2\beta\right)R_{mjip}R_{nklp}+\\
&+\left(\frac{23}{180}+2\beta\right)R_{ijlp}R_{nkmp}+\left(-\frac{7}{90}-\beta\right)R_{iljp}R_{nkmp}-\\
&-\frac{2}{45}R_{mjkp}R_{nlip}-\frac{2}{45}R_{mkjp}R_{nlip}+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{72}-\alpha\right) R_{mikp}R_{nljp}+\left(\frac{29}{360}+\frac{2\alpha}{5}+\frac{7\beta}{5}\right) R_{mkip}R_{nljp}+\\
&+\left(\frac{7}{120}+2\beta\right) R_{mijp}R_{nlkp}+\left(-\frac{7}{90}-\beta\right) R_{mjip}R_{nlkp}+\\
&+\frac{1}{36}R_{ijkp}R_{nlmp}+\frac{1}{36}R_{ikjp}R_{nlmp}+\\
&+\left(-\frac{11}{180}+\frac{9\alpha}{5}-\frac{6\beta}{5}\right) R_{jklp}R_{nmip}+\left(\frac{11}{90}-\frac{18\alpha}{5}+\frac{12\beta}{5}\right) R_{jlkp}R_{nmip}+\\
&+\left(\frac{13}{60}-\frac{3\alpha}{5}+\frac{17\beta}{5}\right) R_{iklp}R_{nmjp}+\left(-\frac{17}{180}+\frac{3\alpha}{5}-\frac{7\beta}{5}\right) R_{ilkp}R_{nmjp}+\\
&+\left(\frac{7}{120}+2\beta\right) R_{ijlp}R_{nmkp}+\left(\frac{7}{36}-\beta\right) R_{iljp}R_{nmkp}-\\
&-\frac{1}{18}R_{ijkp}R_{nmlp}-\frac{7}{72}R_{ikjp}R_{nm,l,p}
\end{aligned}
\qquad (B.6)
$$

$$
\begin{aligned}
T^3_{nmiklj} =& \left(-\frac{47}{120}-\frac{\alpha}{5}-\frac{6\beta}{5}\right) R_{mikp}R_{njlp}+\left(\frac{47}{360}+\frac{2\alpha}{5}+\frac{12\beta}{5}\right) R_{mkip}R_{njlp}-\\
&-\frac{5}{144}R_{iklp}R_{njmp}-\frac{1}{72}R_{ilkp}R_{njmp}+\\
&+\left(\frac{17}{36}+\beta\right) R_{mijp}R_{nklp}+\left(-\frac{17}{180}-2\beta\right) R_{mjip}R_{nklp}-\\
&-\frac{5}{144}R_{ijlp}R_{nkmp}+\frac{7}{144}R_{iljp}R_{nkmp}+\\
&+\left(-\frac{1}{180}+\frac{\beta}{2}\right) R_{mikp}R_{nljp}+\left(-\frac{47}{720}-\frac{\alpha}{5}-\frac{6\beta}{5}\right) R_{mkip}R_{nljp}+\\
&+\left(-\frac{19}{360}-\frac{\beta}{2}\right) R_{mijp}R_{nlkp}+\left(\frac{17}{360}+\beta\right) R_{mjip}R_{nlkp}-\\
&-\frac{1}{72}R_{iklp}R_{nmjp}-\frac{1}{72}R_{ilkp}R_{nmjp}+\\
&+\frac{7}{144}R_{ijlp}R_{nmkp}-\frac{11}{144}R_{iljp}R_{nmkp}\,.
\end{aligned}
\qquad (B.7)
$$

It would be interesting to know if there is a solution that can be written in an elegant compact form that may suggest a generalization to higher orders.

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